1 Poincaré duality

Def An orientation on an n dim'l $\mathbb{R}$-vector space V is a choice of a connected component of $\Lambda^n V - \{0\}$. An ordered basis $v_1, \dots v_n$ is positively oriented if $v_1 \wedge \dots v_n$ is in the given compont

Observe that an orientation is determined by a nonzero fctn $\omega : \Lambda^n V \to \mathbb{R}$. Namely $\omega^{-1}(\mathbb{R}_+)$ gives an orientation. With this in mind:

Def An n dim'l C^∞ manifold X is orientable if there exists a nowhere zero n-form ω on X.
Two such forms ω_i determine the same orientation if $\omega_2 = f\,\omega_1$, where $f > 0$ everywhere.

<u>Ex</u> A complex manifold X is orientable. If $z_1 = x_1 + iy_1, \ldots z_n = x_n + iy_n$ are holomorphic coordinates,

$$\omega = f \, dx_1 \wedge dy_1 \wedge dx_2 \wedge \cdots$$

with $f > 0$, gives an orientation.

<u>Ex</u> The Mobius strip, etc. are well known to be nonorientable.

<u>Reminders</u> Let $\mathcal{E}^p(X)$ = space of p-forms

We have operators

$$d : \mathcal{E}^p(X) \to \mathcal{E}^{p+1}(X)$$

Such that $d^2 = 0$. We defined

de Rham cohomology

$$H^p_{dR}(X) = \frac{\ker d : \mathcal{E}^p(X) \to \mathcal{E}^{p+1}(X)}{\operatorname{im} d : \mathcal{E}^{p-1}(X) \to \mathcal{E}^p(X)}$$

Recall de Rham's thm:

$$H^p_{dR}(X) \cong \underbrace{H^p(X, \mathbb{R}_X)}_{\text{sheaf cohomology}}$$

We have a stronger form

<u>Thm</u> (multiplicative de Rham's thm)
Under the previous isomorphism, cup product on the right, corresponds to $\wedge$ on the left.

The final thing we need to recall is integration on manifolds. If X is a compact oriented manifold, then there is an operator

$$\int_X : \Sigma^n(X) \longrightarrow \mathbb{R}$$

such that Stokes' theorem holds

i.e.
$$\int_X d\alpha = 0$$

[Our manifolds don't have a boundary which is why we get 0]

Thm (Poincaré duality I)
If X is a compact connected orientable n dimensional manifold, then we have an isomorphism

$$H^n_{dR}(X) \cong \mathbb{R}$$

given by $\alpha \mapsto \int_X \alpha$.
For any i,

$$H^i_{dR}(X) \times H^{n-i}_{dR}(X) \to \mathbb{R}$$
$$(\alpha, \beta) \longmapsto \int_X \alpha \wedge \beta$$

gives a nondegenerate pairing.
Hence
$$H^{n-i}_{dR}(X) \cong H^i_{dR}(X)^*$$
$$\left(\cong H^i_{dR}(X) \text{ noncanonically}\right).$$

Ex Let $X = \mathbb{R}^n / \mathbb{Z}^n$ be the n-torus. Then

$$H^i_{dR}(X) \cong \Lambda^i \mathbb{R}^n$$

Numerically, Poincaré reduces to the well known identity $\binom{n}{i} = \binom{n}{n-i}$.

2 Proof of Poincaré

Given a possibly noncompact manifold X, let

$$\Sigma^p_c(X) = \{ \alpha \in \Sigma^p(X) \mid \alpha \text{ has compact support} \}.$$

Clearly $d(\Sigma^p_c(X)) \subset \Sigma^{p+1}_c(X)$

So $\Sigma^\bullet_c(X) \subset \Sigma^\bullet(X)$ is a subcomplex

Define compactly supported de Rham

cohomology by

$$H^p_{cdR}(X) = H^p(\Sigma^\bullet_c(X), d)$$

The one computation we need is

<u>Thm</u> $H^p_{cdR}(\mathbb{R}^n) = \begin{cases} \mathbb{R} & \text{if } p = n \\ 0 & \text{otherwise} \end{cases}$

(See Spivak's Diff geom vol I
or ...)

We'll prove a statement both weaker and stronger than before

<u>Thm</u> If X is orientable connected n-manifold, then

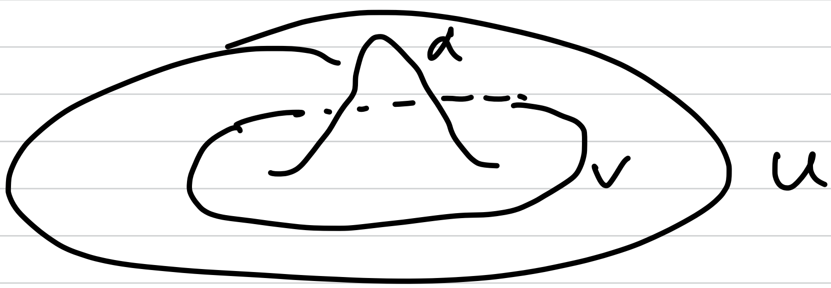
$$H^{n-i}_{cdR}(X) \cong H^i_{dR}(X)^*$$

Define $$\mathcal{C}^p(U) = \mathrm{Hom}_{\mathbb{R}}(\Sigma_c^p(U), \mathbb{R})$$

If $U \supseteq V$, and $\beta \in \mathcal{C}^p(U)$

$\alpha \in \Sigma_c^p(V)$, define

$$\beta|_V(\alpha) = \beta(\alpha \text{ extended by } 0 \text{ to } U)$$



This makes $\mathcal{C}^p$ a presheaf.

In fact

<u>Lemma</u> $\mathcal{C}^p$ is a sheaf.

Define $\delta : \mathcal{C}^p \longrightarrow \mathcal{C}^{p+1}$

by $(\delta\beta)(\alpha) = \underbrace{(-1)^{p+1}}_{\text{can ignore}} \beta(d\alpha)$

$\alpha \in \mathcal{E}_c^{p+1}(\mathcal{U})$

Then $\delta^2 = 0$, so we have a complex of sheaves

Define

$$\mathbb{R}_X \longrightarrow \mathcal{C}^0$$
$$1 \longmapsto \int_X$$

Putting previous results together shows

Lemma on $\mathbb{R}_X \to \mathcal{C}^0 \xrightarrow{\delta} \mathcal{C}^1 \xrightarrow{\delta} \cdots$

is a soft resolution.

pf of Poincaré duality (2nd version)

On the one hand

$$H^p(X, \mathbb{R}_X) \cong H^p_{dR}(X)$$

On the other hand

$$H^p(X, \mathbb{R}) = H^p(\mathcal{T}^\bullet(X))$$
$$\cong H^{n-p}_{cdR}(X)^*$$

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3 Riemann Surfaces

Prop If X is a compact (always connected) Riemann surface, then $b_1(X) = \dim H^1(X, \mathbb{R})$ is even.

$b_0(X) = b_2(X) = 1$, and all other Betti numbers are zero

proof The pairing

$$\langle \alpha, \beta \rangle = \int_X \alpha \wedge \beta$$

$$H^1(X) \times H^1(X) \longrightarrow \mathbb{R}$$

is nondegenerate by Poincaré duality and skew symmetric.

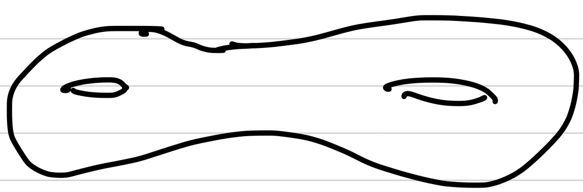
So now use

lemma If A is an $n \times n$ skew symmetric matrix, with n odd, then $\det(A) = 0$.

$H^0(X) = \{ f \in C^\infty(X) \mid df = 0 \} = \mathbb{R}$

and other values follow from Poincaré //

Def The genus of a compact R.S. X is $g = \frac{1}{2} \dim H^1(X)$

e.g. $g = 2$ for 

A useful observation is

Cor (of thm) The Euler characteristic

$$e(X) = \sum (-1)^i b_i(X) = 2 - 2g$$

Next we want to compute some holomorphic invariants

Let $\mathcal{O}_X$ = sheaf of holomorphic functions

$$\subset C^\infty(U) \otimes_{\mathbb{R}} \mathbb{C} =: C^\infty_{\mathbb{C}}(U)$$

Given a local holomorphic coordinate z on U, let

$$x = \mathrm{Re}\, z \quad \text{and} \quad y = \mathrm{Im}\, z.$$

$$dz = dx + i\,dy \in \mathcal{E}^1_{\mathbb{C}}(U) := \mathcal{E}^1(U) \otimes_{\mathbb{R}} \mathbb{C}$$

and $$d\bar{z} = dx - i\,dy \in \mathcal{E}^1_{\mathbb{C}}(U)$$

$$\mathcal{E}^{1,0}(U) = C^\infty_{\mathbb{C}}(U) \cdot dz \,;\quad \mathcal{E}^{0,1}(U) = C^\infty_{\mathbb{C}}(U)\, d\bar{z}$$

We extend $d : C^\infty_{X,\mathbb{C}} \to \mathcal{E}^1_{X,\mathbb{C}}$ $\mathbb{C}$-linearly. We split

$d = \partial + \bar{\partial}$, where

$$\partial f = \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right) dz \in \mathcal{E}^{1,0}$$

$$\bar{\partial} f = \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right) d\bar{z} \in \mathcal{E}^{0,1}$$

Prop f is holomorphic $\iff \bar{\partial} f = 0$

pf This follows from the Cauchy-Riemann eq. //

lemma $d(\mathcal{O}_X) \subset \Omega^1_X$

pf Obvious. //

We define $\partial : \mathcal{E}^{1,0} \to \mathcal{E}^2_{\mathbb{C}}$, $\bar{\partial} : \mathcal{E}^{0,1} \to \mathcal{E}^2_{\mathbb{C}}$ by

$$\bar{\partial}(f\,dz) = (\bar{\partial} f) \wedge dz$$
$$\partial(f\,d\bar{z}) = (\partial f) \wedge d\bar{z}$$

We again have

$$d = \partial + \bar{\partial}$$

We note $\mathcal{E}^2_{\mathbb{C}} = C^\infty_{\mathbb{C}} \cdot dz \wedge d\bar{z} =: \mathcal{E}^{11}$

Thm If $D \subset \mathbb{C}$ is an open disk,

$$\mathcal{O}(D) \xrightarrow{d} \Omega^1_X(D)$$

is surjective.

Pf Suppose $\alpha \in \Omega^1(D)$, then

$$d\alpha = (\partial + \bar\partial)\alpha = \bar\partial \alpha = 0$$

Therefore

$$\alpha = df$$

by the Poincaré lemma

However, we must have $\bar\partial f = 0$.

Therefore $f \in \mathcal{O}(D)$ //

Cor We have an exact sequence

$$0 \to \mathbb{C}_X \to \mathcal{O}_X \xrightarrow{d} \Omega^1_X \to 0$$

It follows that we get an exact seq.

$$
\begin{tikzcd}
0 \arrow[r] & \underbrace{H^0(X, \mathbb{C}_X)}_{\mathbb{C}} \arrow[r, "\sim"] & \underbrace{H^0(X, \mathcal{O}_X)}_{\mathbb{C}} \arrow[dll, "d"] \\
H^0(X, \Omega^1_X) \arrow[r] & H^1(X, \mathbb{C}) \arrow[r] & H^1(X, \mathcal{O}_X) \arrow[dll, "d"] \\
H^1(X, \Omega^1_X) \arrow[r] & \underbrace{H^2(X, \mathbb{C})}_{\shortparallel \atop \mathbb{C} \text{ by}}
\end{tikzcd}
$$

Since $H^0(X, \mathcal{O}_X) = \mathbb{C}$, it follows that the first map labelled d is zero. The second map labelled d is also zero by

Thm $\quad H^1(X, \Omega^1_X) \cong \mathbb{C}$

It follows that

Prop There is a short exact seq.

$$0 \to H^0(X, \Omega^1_X) \to H^1(X, \mathbb{C}) \to H^1(X, \mathcal{O}_X) \to 0$$

Cor $\quad b_1(X) = \dim H^0(\Omega^1_X) + \dim H^1(\mathcal{O}_X)$

4 Serre duality

Thm (Serre duality) Let X be a compact Riemann surface, and L be a line bundle on L, cup product gives a nondegenerate pairing

$$H^1(X,L) \times H^0(X, \Omega^1_X \otimes L^*) \to H^1(X, \Omega^1_X) \cong \mathbb{C}$$

Therefore, there is a natural isomorphism

$$H^1(X, L) \cong H^0(X, \Omega^1_X \otimes L^*)^*$$

Cor The genus

$$g(X) = \dim H^0(X, \Omega^1_X) = \dim H^1(X, \mathcal{O}_X)$$

pf Serre implies

$$\dim H^0(X, \Omega^1) = \dim H^1(X, \mathcal{O}_X)$$

The result follows from this plus the previous cor:

$$2g = \dim H^0(\Omega^1_X) + \dim H^1(\mathcal{O}_X)$$

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5 Harmonic Forms

We will say a few words about proofs. First, we recall that holomorphic functions and harmonic functions are closely related. This can be explained by noting that

$$\partial \bar{\partial} f = \frac{1}{4} \underbrace{\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right)}_{\Delta} f \, dz \wedge d\bar{z}$$

$$\bar{\partial} \partial f = - \partial \bar{\partial} f$$

Therefore

f holomorphic $\Rightarrow$ f & $\bar{f}$ harmonic

$\Rightarrow$ $\mathrm{Re} f$ harmonic

Conversely, it's known that a real valued harmonic function is the real part of a holomorphic function locally.

This suggests

Def A 1-form α a R.S. is harmonic if its a linear combination of holomorphic 1-form and the complex conjugate of a holomorphic 1-form (= antiholomorphic 1-form).

Rmk This isn't the usual def, but it's equivalent

lemma A harmonic 1-form is closed i.e. it satisfies $d\alpha = 0$.

pf when α is holomorphic

$$d\alpha = \bar{\partial}\alpha = 0$$

Likewise for the antiholomorphic case //

"Hodge" Thm (Weyl) When X is a compact R.S. every class in $H^1(X, \mathbb{C})$ has a unique harmonic representation.

Historical Note Hodge proved the analogous thm in higher dims.