1 Morphisms of Sheaves.

Def Given a topological space X, and presheaves $\mathcal{F}$ and $\mathcal{G}$ of sets, groups... a **morphism** $\eta : \mathcal{F} \to \mathcal{G}$ is a collection of maps, homomorphisms...

$$\eta_U : \mathcal{F}(U) \to \mathcal{G}(U)$$

compatible with restriction in the sense that

$$\begin{tikzcd} \mathcal{F}(U) \arrow[r, "\eta_U"] \arrow[d, "\rho_{UV}"'] & \mathcal{G}(U) \arrow[d, "\rho_{UV}"] \\ \mathcal{F}(V) \arrow[r] & \mathcal{G}(V) \end{tikzcd}$$

commutes $\forall\, V \subset U$.

A **morphism** of sheaves is defined the same way.

Ex Let $\mathcal{F}$ be a sheaf of (comm) rings on X, such as C^∞_X, $\mathcal{O}_X^{an}$, $\mathcal{O}_X$ and let $f \in \mathcal{O}(X)$ be a global section

Consider the map

$$\mathcal{F}(U) \xrightarrow{f|_U \cdot} \mathcal{F}(U)$$

given by multiplication by $f|_U$.

This is a morphism of sheaves of abelian groups (Check!)

Ex 2 Let $d: C^\infty_{\mathbb{R}} \to C^\infty_{\mathbb{R}}$ be given by $d(f) = f'$

This is a morphism of sheaves of abelian groups

Ex 3 Let C_X be the sheaf of continuous real valued functions on a manifold X. Then the inclusions $C^\infty_X(U) \subseteq C_X(U)$ give a morphism $C^\infty_X \to C_X$ of sheaves of rings.

We say C^∞_X is a subsheaf of C_X

The collection of presheaves on X and morphisms forms a category $PSh(X)$ and sheaves form a full subcategory

$$Sh(X) \subset PSh(X)$$

2 Stalk/sheafification

Given a presheaf $\mathcal{F}$ on X and a point $p \in X$, two sections $f_1 \in \mathcal{F}(U_1)$ and $f_2 \in \mathcal{F}(U_2)$ defined on nbhds of p have the same germ at p if $f_1|_{U_3} = f_2|_{U_3}$ for $p \in U_3 \subset U_1 \cap U_2$. This forms an equivalence relation and the set of equivalence classes is called the stalk $\mathcal{F}_p$

Alternatively

$$\mathcal{F}_x = \varinjlim_{p \in \mathcal{U}} \mathcal{F}(\mathcal{U})$$

* Discuss examples $\operatorname{Spec} A$, $(\mathbb{C}, \mathcal{O}^{an}_{\mathbb{C}})$

lemma The stalk is a functor from $PSh(X) \to Sets$.

Thm/Def Given a presheaf $\mathcal{F}$, there is a sheaf $\mathcal{F}^+$ and a morphism $\mathcal{F} \to \mathcal{F}^+$ such that for any morphism

$$\mathcal{F} \to \mathcal{G}$$

where $\mathcal{G}$ is a sheaf, there exists a unique comm. diagram

$$\begin{tikzcd} \mathcal{F} \arrow[r] \arrow[dr] & \mathcal{F}^+ \arrow[d, dashed] \\ & \mathcal{G} \end{tikzcd}$$

$\mathcal{F}^+$, which is unique up to iso, is called the sheafification of $\mathcal{F}$

Furthermore $\mathcal{F}_p \cong \mathcal{F}_p^+$

Sketch

We just give the construction and an outline. The basic model is $\mathcal{F}$ = presheaf of constant $\mathbb{R}$-valued functions.

Then $\mathcal{F}^+$ = sheaf of locally constant functions.

In general replace $\mathbb{R}$ by the disjoint union of stalks, $F = \bigcup_{p \in X} \mathcal{F}_p$

Let $\mathcal{F}^+(U) = \{ f : U \to F \mid$
$\forall p \in U, \exists p \in V \subseteq U$ and $s \in \mathcal{F}(V)$
s.t. $f(q) = s_q$, the germ of s at $q \}$

We have a morphism

$$\mathcal{F} \to \mathcal{F}^+$$

send $s \in \mathcal{F}(U) \longmapsto (s_p)_{p \in U} \in \mathcal{F}^+(U)$.

3 Sheaves of Abelian Groups

We will mainly be interested in sheaves abelian groups (with possible extra structure). Let $Ab(X)$ denote the category of these on X. Let us also study the bigger category of presheaves of abelian groups $PAb(X)$.

$PAb(X)$ is an abelian category. We won't define this precisely but roughly it means that the standard constructions and properties from the category of abelian groups Ab carry over:

1) $\mathrm{Hom}_{\mathrm{PAb}}(\mathcal{F}, \mathcal{G})$ is naturally an abelian group.

2) We have direct sums

$$(\mathcal{F} \oplus \mathcal{G})(U) = \mathcal{F}(U) \oplus \mathcal{G}(U)$$

3) Given a morphism

$$\eta : \mathcal{F} \longrightarrow \mathcal{G}$$

in $\mathrm{PAb}(X)$, we can form the presheaf kernel

$$p\ker \eta\,(U) = \ker \eta_U \subset \mathcal{F}(U)$$

and presheaf image

$$p\,\mathrm{im}\,\eta\,(U) = \mathrm{im}\,\eta_U \subset \mathcal{G}(U).$$

4) We have exact sequences

$$\cdots\ \mathcal{F}^i \xrightarrow{\eta^i} \mathcal{F}^{i+1} \longrightarrow \cdots$$

This means $p\ker \eta^{i+1} = p\operatorname{Im} \eta^i \quad \forall i$

i.e. $\forall\, U$

$$\cdots\ \mathcal{F}^i(U) \to \mathcal{F}^{i+1}(U) \to \cdots$$

is exact in the usual sense.

Let us now turn to $Ab(X)$.

$Ab(X)$ is also abelian. However it's more subtle.

Points (1) & (2) carry over without any problem, and half of (3) is OK.

<u>Lemma</u> If $\eta : \mathcal{F} \to \mathcal{G}$ is a morphism in $Ab(X)$, then $p\ker \eta$ is a sheaf.

<u>Pf</u> Let $\underline{K} = p\ker \eta$. This a subpresheaf of $\mathcal{F}$. Given a cover $\{U_i\}$ of U and compatible sections $k_i \in \underline{K}(U_i)$ (compatible means $k_i|_{U_i \cap U_j} = k_j|_{U_i \cap U_j}$).

We have a unique $k \in \mathcal{F}(U)$ s.t. $k|_{U_i} = k_i$ because $\mathcal{F}$ is a sheaf. Then

$$\eta(k)|_{U_i} = \eta(k|_{U_i}) = 0$$

Therefore $\eta(k) = 0$ $\because$ $\mathcal{G}$ is a sheaf.

Therefore $k \in \underline{K}(U)$ //.

On the other hand pim need not be a sheaf.

<u>Exa.</u> Let $S^1 = \mathbb{R}/\mathbb{Z}$ be the circle.

Consider $d : C^\infty_{S^1} \to C^\infty_{S^1}$ be the derivative. Check

$$1 \notin \operatorname{pim}(d) : C^\infty(S^1) \to C^\infty(S^1)$$

but $1|_{U_i} \in C^\infty(U_i)$ for the cover

$U_1 = (0,1) \bmod \mathbb{Z}$

$U_2 = (1/2, 3/2) \bmod \mathbb{Z}$

Given a morphism

$$\eta : \mathcal{F} \to \mathcal{G}.$$

in $Ab(X)$, we define the sheaf kernel and image by

$$\begin{cases} \ker \eta = p\ker \eta \\ \operatorname{im} \eta = (p\operatorname{im} \eta)^{+} \end{cases}$$

We define a sequence of sheaves

$$\cdots \mathcal{F}^{i} \xrightarrow{\eta^{i}} \mathcal{F}^{i+1} \xrightarrow{\eta^{i+1}} \cdots$$

to be exact if

$$\ker \eta^{i+1} = \operatorname{im} \eta^{i}$$

lemma A sequence of sheaves

$$\cdots \mathcal{F}^{i} \xrightarrow{\eta^{i}} \mathcal{F}^{i+1} \longrightarrow \cdots$$

is exact $\iff$ $\forall\, p \in X$

$$\cdots \mathcal{F}^{i}_{p} \longrightarrow \mathcal{F}^{i+1}_{p} \longrightarrow \cdots$$

is exact in the usual sense.

It's enough to prove inclusion $\ker \eta^{i+1} = \operatorname{im} \eta^i$ is an iso. For this we need 11

Sublemma If $\eta : \mathcal{F} \to \mathcal{G}$ is a morphism of sheaves, η is an isomorphism iff $\forall p$, $\eta_p : \mathcal{F}_p \xrightarrow{\sim} \mathcal{G}_p$

Remark: This is False for presheaves. (e.g. consider $\mathcal{F} \to \mathcal{F}^+$, when $\mathcal{F}$ is not a sheaf.)

pf of sublemma We just need to show

$\forall p$ η_p an iso $\Rightarrow$ $\forall \mathcal{U}$, $\eta_\mathcal{U}$ is an iso.

Suppose $\eta(f_1) = \eta(f_2)$. Then

$\forall p$, $f_{1p} = f_{2p}$ $\Rightarrow$ $\exists$ open cov $\{\mathcal{U}_i\}$ of $\mathcal{U}$

s.t. $f_1|_{\mathcal{U}_i} = f_2|_{\mathcal{U}_i}$ $\Rightarrow$ $f_1 = f_2$ by sheaf property.

Therefore η_u is injective.

Suppose $g \in \mathcal{G}(\mathcal{U})$. Then $\forall p$ g_p lies in the image of η_p. Therefore $g|_{u_i} = \eta(f_i) \in \mathcal{F}(\mathcal{U}_i)$. By the first half f_i are unique. Therefore they patch to a section $f \in \mathcal{F}($ s.t. $g = \eta(f)$

We come to an important and subtle point: An exact sequence in $Ab(X)$ need not be exact in $PAb(X)$.

Ex let $X = S^1 = \mathbb{R}/\mathbb{Z}$

Consider the sequence

$$0 \to \mathbb{R}_X \to C^\infty_X \xrightarrow{d} C^\infty_X \to 0$$

We saw that $p\,\mathrm{im}\,d \neq C^\infty_X$

so it's not exact in $PAb(X)$

However, we claim that it is an exact sequence of sheaves.

Since it's enough to check exactness on stalks, it suffices to do it on an interval.

But exactness of

$$0 \to \mathbb{R} \to C^\infty(a,b) \xrightarrow{d} C^\infty(a,b) \to 0$$

is clear by the fundamental thm of calculus.

We will see many other examples later. Here is the key point

Theorem If

$$0 \to \mathcal{A} \to \mathcal{B} \xrightarrow{\eta} \mathcal{C} \to 0$$

is an exact sequence in $Ab(X)$

Then

$$0 \to \mathcal{A} \to \mathcal{B} \to \mathcal{C}$$

is exact in $PAb(X)$

[This says that $Ab(X) \subset PAb(X)$ is left exact.]

proof We can identify

$$\mathcal{A} = \ker \eta = p\ker \eta.$$

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Define $\Gamma(X, -) : Ab(X) \to Ab$

by $\Gamma(X, \mathcal{F}) = \mathcal{F}(X)$

Cor $\Gamma(X, -)$ is left exact

The previous example shows that $\Gamma(X, -)$ is usually not exact.

4 Projective Space

Let's take a break from sheaf theory, and discuss an important example. Let k be a field.

Projective space of dim n is

$$\mathbb{P}^n_k = \{ L \subset k^{n+1} \mid L \text{ is a } 1 \text{ dim'l subspace} \}$$

$$= k^{n+1} \setminus \{0\} \big/ v \sim \lambda v, \quad \lambda \in k^*$$

When $k = \mathbb{R}$ or $\mathbb{C}$, k^{n+1} has a Euclidean topology, and we give $\mathbb{P}^n_k$ the quotient topology.

For k arbitrary, we can also use the Zariski topology on k^{n+1} and take the quotient.

Let $[x_0, \dots x_n]$ = equiv class of $(x_0, \dots x_n) \in k^{n+1}$

Set $\mathcal{U}_i = \{[x_0, \dots x_n] \mid x_i = 1\}$

This is an open covering in any of the above topologies, and we have a homeomorphism

$$k^n \cong \mathcal{U}_i$$

$$(y_1, \dots y_n) \longmapsto (y_1, \dots y_{i-1}, 1, y_i, \dots)$$

In particular, when $k = \mathbb{R}$, $\mathbb{P}^n_{\mathbb{R}}$ is a topological n-manifold, and $\mathbb{P}^n_{\mathbb{C}}$ is a topological $2n$-manifold.

We can make $X = \mathbb{P}^n_{\mathbb{R}}$ (and $\mathbb{P}^n_{\mathbb{C}}$) into C^∞-manifolds by declaring

$$f \in C^\infty_X(\mathcal{U}) \iff f \circ \pi \text{ is } C^\infty$$

on $\pi^{-1}\mathcal{U} \subset \mathbb{R}^{n+1}$

where π is the projection

When $k = \bar{k}$, we define

$f \in \mathcal{O}_{\mathbb{P}^n_k}(U) \iff f \circ \pi$ is regular.

Lemma $(\mathbb{P}^n_k, \mathcal{O}_{\mathbb{P}^n})$ is a prevariety (and in fact a variety which means $\Delta \subset \mathbb{P}^n \times \mathbb{P}^n$ is closed)

Finally, consider $X = \mathbb{P}^n_{\mathbb{C}}$.

define
$$f \in \mathcal{O}^{an}_{\mathbb{P}^n_{\mathbb{C}}}(U) \iff f \circ \pi \text{ is holomorphic}$$
(when a function of several variables is holomorphic if it be expanded as a convergent power series about every pt.)

Lemma $(\mathbb{P}^1_{\mathbb{C}}, \mathcal{O}^{an}_{\mathbb{P}^1_{\mathbb{C}}})$ is a Riemann surface. [This example should be familiar since it's the Riemann sphere.