1. Sheaf Cohomology

Last time we defined

$$G(\mathcal{F}) = \bigoplus_{p \in X} \mathcal{F}_p ,$$

$$C(\mathcal{F}) = \left(G(\mathcal{F}) / \mathcal{F} \right)^+$$

$$H^0(X, \mathcal{F}) = \Gamma(X, \mathcal{F})$$

$$H^1(X, \mathcal{F}) = \operatorname{coker}\left[\Gamma(G(\mathcal{F})) \to \Gamma(C(\mathcal{F})) \right]$$

$$H^{n+1}(X, \mathcal{F}) = H^1(X, C^n(\mathcal{F}))$$

But it remains to establish the basic properties.

The first thing we need is the

<u>**Snake Lemma**</u> Given a commutative diagram in $Ab(X)$ with exact rows

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] \arrow[d, "\alpha"'] & B \arrow[r, "f"] \arrow[d, "\beta"'] & C \arrow[r] \arrow[d, "\gamma"'] & 0 \\
0 \arrow[r] & D \arrow[r] & E \arrow[r, "g"'] & F &
\end{tikzcd}$$

There exist an exact seq.

$$0 \to \ker\alpha \to \ker\beta \to \ker\gamma \xrightarrow{\delta} \operatorname{coker}\alpha \to \operatorname{coker}\beta \to \operatorname{coker}\gamma \quad (*)$$

Furthermore the last map is surjective if g is.

<u>Sketch</u> The only map which is not obvious is δ. This is typically defined using a diagram chase, but this won't generalize well to sheaves, so we need to be more careful. Let $P = f^{-1}(\ker\gamma) \subset B$ and $K = \ker(f) \cap P$. Then

One checks that

$$P/K \cong \ker\gamma$$

$g \circ \beta(P) = 0 \Rightarrow \exists\, \delta'$ as depicted

$$\begin{tikzcd}
 & & P \arrow[dl, dashed, bend right, "\delta'"'] \arrow[d, "\beta"] \arrow[dr, bend left, "0"] & & \\
0 \arrow[r] & D \arrow[r] & E \arrow[r, "g"'] & F \arrow[r] & 0
\end{tikzcd}$$

Furthermore one checks $\delta'(K) \subset \operatorname{im} \alpha$ so we get an induced map

$$\operatorname{Ker} \gamma \cong P/K \longrightarrow \operatorname{coker} \alpha$$

To finish, we need to check exactness of (*), but we can do this on stalks, so the usual pf for Ab works. //

Lemma 2 The functor $G : Ab(X) \to Ab(X)$ is exact.

Pf If

$$0 \to \mathcal{A} \to \mathcal{B} \to \mathcal{C} \to 0$$

is exact in $Ab(X)$

$$0 \to \mathcal{A}_p \to \mathcal{B}_p \to \mathcal{C}_p \to 0$$

is exact. Thus

$$0 \to G(\mathcal{A}) \to G(\mathcal{B}) \to G(\mathcal{C}) \to 0$$

is exact. //

Lemma 3 If $0 \to \mathcal{A} \to \mathcal{B} \to \mathcal{C} \to 0$ is exact in $Ab(X)$, then

$$\begin{array}{ccccccccc}
0 \to & G(\mathcal{A}) & \to & G(\mathcal{B}) & \to & G(\mathcal{C}) & \to 0 \\
& \downarrow & & \downarrow & & \downarrow & \\
0 \to & C(\mathcal{A}) & \to & C(\mathcal{B}) & \to & C(\mathcal{C}) & \to 0
\end{array}$$

has exact rows.

Pf Apply the snake lemma to

$$\begin{array}{ccccccccc}
0 \to & \mathcal{A} & \to & \mathcal{B} & \to & \mathcal{C} & \to 0 \\
& \downarrow & & \downarrow & & \downarrow & \\
0 \to & G(\mathcal{A}) & \to & G(\mathcal{B}) & \to & G(\mathcal{C}) & \to 0
\end{array}$$

//

Thm Given a short exact seq^n

$$0 \to A \to B \to C \to 0$$

in $Ab(X)$, there is a long exact sequence

$$0 \to H^0(X, A) \to H^0(X, B) \to H^0(X, C) \to H^1(X, A) \to \cdots$$

Pf We just prove exactness of the first 6 terms.

Apply lemma 3 and use the fact that $G(\mathcal{A})$ is flasque & a theorem from last week to obtain

$$
\begin{tikzcd}
0 \arrow[r] & \Gamma(G(\mathcal{A})) \arrow[r] \arrow[d] & \Gamma(G(\mathcal{B})) \arrow[r] \arrow[d] & \Gamma(G(\mathcal{C})) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \Gamma(C(\mathcal{A})) \arrow[r] & \Gamma(C(\mathcal{B})) \arrow[r] & \Gamma(C(\mathcal{C})) &
\end{tikzcd}
$$

Now apply the snake lemma to this to get the 6 term sequence claimed in the theorem //

Cor If $H^1(X, \mathcal{A}) = 0$, then

$$\Gamma(X, \mathcal{B}) \to \Gamma(X, \mathcal{C})$$

is surjective.

To use this we need to establish a so called vanishing theorem. Here is a simple example

Prop If $\mathcal{F}$ is flasque, $\forall\, i > 0$ $H^i(X, \mathcal{F}) = 0$.

Pf Since $\mathcal{F}$ is flasque, we have an exact seq

$$0 \to H^0(U, \mathcal{F}) \to H^0(U, \mathcal{G}(\mathcal{F})) \to H^0(U, \mathcal{C}(\mathcal{F})) \to 0$$

So in particular, $H^1(X, \mathcal{F}) = 0$.

Also we have surjections (indicated with $\twoheadrightarrow$)

$$\begin{tikzcd}
\Gamma(X, \mathcal{G}(\mathcal{F})) \arrow[r, two heads] \arrow[d, two heads] & \Gamma(U, \mathcal{G}(\mathcal{F})) \arrow[d, two heads] \\
\Gamma(X, \mathcal{C}(\mathcal{F})) \arrow[r] & \Gamma(U, \mathcal{C}(\mathcal{F}))
\end{tikzcd}$$

which implies $\mathcal{C}(\mathcal{F})$ is flasque. Therefore

$$H^2(X, \mathcal{F}) = H^1(X, \mathcal{C}(\mathcal{F})) = 0$$

etc. //

2 Soft Sheaves

One problem with flasque sheaves is that they don't occur "in nature". We introduce a related class which does.

on.

Given a sheaf $\mathcal{F}$ and a closed set Y,
let $\mathcal{F}(Y) = \varinjlim_{U} \mathcal{F}(U)$

as U runs over open nbhd of Y.

Def A sheaf $\mathcal{F}$ on X is **soft** if $\forall$ closed Y
the restriction $\mathcal{F}(X) \to \mathcal{F}(Y)$
is surjective.

Although the definition is similar to flasque
(and in fact flasque $\Rightarrow$ soft), there are
many more examples.

Thm If X is a metrizable space, then
the sheaf continuous $\mathbb{R}$-valued functions
is soft.

pf: Given a closed Y, by point set
topology (Urysohn's lemma),

$\exists$ a continuous $f : X \to \mathbb{R}$ which is 1 in given a nbhd of Y and 0 away from Y. Given $f \in C(Y)$ φf can be extended by 0 to a global continuous function.

Thm If X is a C^∞-manifold, C^∞_X is soft.

pf By standard tricks, we can choose a C^∞ cutoff function as above.

Def A sheaf $\mathcal{F} \in Ab(X)$ is a *sheaf of modules* over a sheaf of rings $\mathcal{R}$, if each $\mathcal{F}(U)$ is an $\mathcal{R}(U)$-module and $(rf)|_V = r|_V \cdot f|_V$ for $r \in \mathcal{R}(U)$, $f \in \mathcal{F}(U)$.

Prop (Over a metric space X.) If $\mathcal{R}$ is a soft sheaf of rings, then any $\mathcal{R}$-module is soft.

Pf Choose $f \in \mathcal{F}(Y)$, with Y closed.

Then f is defined on an open nbhd $\mathcal{U} \supset Y$.

Since $\mathcal{R}$ is soft, the section $= 1$ on Y and 0 on $X - \mathcal{U}$, extends to a global section $\rho \in \mathcal{R}(X)$. Then $\rho \cdot f$ extends by 0 to all of X. //

Cor If X is a manifold, any module over C^∞_X is soft.

Enough examples, now we come to the key point.

Main Thm If X is metrizable, and $\mathcal{F} \in Ab(X)$ is soft, then

$$H^i(X, \mathcal{F}) = 0 \quad \forall\, i > 0.$$

The pf is similar to what we did for flasque sheaves. The first step is

Thm If X is metrizable, and

$$0 \to \mathcal{A} \to \mathcal{B} \to \mathcal{C} \to 0$$

is exact in $Ab(X)$ and $\mathcal{A}$ soft, then

$$\mathcal{B}(X) \to \mathcal{C}(X)$$

is surjective.

pf When $\mathcal{A}$ was flasque, given $\gamma \in \mathcal{C}(X)$, we used Zorn's lemma to choose a maximal open $U \subset X$ s.t. γ lies in the image of $\mathcal{B}(U)$. This won't work for closed sets because unions of closed sets needn't be closed. So we modify the strategy.

A theorem of Stone says that metric spaces are paracompact. This implies that we can choose a locally finite open cover $\{U_i\}$

s.t $\mathcal{U}_i \supset \overline{V}_i$, where V_i is another open cover. Local finiteness means any pt has a nbhd meeting only a finite number of $\mathcal{U}_i$. This condition can be used to show $\bigcup_{i \in J} \overline{V}_i$ is closed for any set J. So now we choose $Y = \bigcup_{i \in J} \overline{V}_i$ to be a maximal set s.t. $\gamma|_Y$ lifts $\mathcal{B}(Y)$. If $Y \neq X$, then we can argue as we did before using softness of $\mathcal{Q}$ that $\beta|_{Y'}$ lifts for some $Y' = Y \cup V_k \supsetneq Y$ //

To finish the pf of the Main thm we need

Lemma

(a) A flasque sheaf is soft

b) If $0 \to \mathcal{A} \to \mathcal{B} \to \mathcal{C} \to 0$ is exact in $Ab(X)$ and $\mathcal{A}$ & $\mathcal{B}$ are soft then so is $\mathcal{C}$.

pf Exercise

pf of Main thm

Consider

$$0 \to \mathcal{A} \to G(\mathcal{A}) \to C(\mathcal{A}) \to 0$$

By previous thm

$$\Gamma(G(\mathcal{A})) \twoheadrightarrow \Gamma(C(\mathcal{A}))$$

So $H^1(\mathcal{A}) = 0$.

By the previous lemma $C(\mathcal{A})$ is soft. Therefore

$$H^2(\mathcal{A}) = H^1(C(\mathcal{A})) = 0$$

etc. //

Cor If X is a C^∞ manifold and $\mathcal{F}$ is a sheaf of modules on C^∞_X, then

$$H^i(X, \mathcal{F}) = 0 \qquad i > 0.$$

We're ready to do our first serious computation

Ex Let $X = S^1 = \mathbb{R}/\mathbb{Z}$. We had an exact seq

$$0 \to \mathbb{R}_X \to C^\infty_X \xrightarrow{d} C^\infty_X \to 0$$

s.t.

$$\Gamma(X, C^\infty) \xrightarrow{d} \Gamma(X, C^\infty)$$

is not surjective. Therefore

$$H^1(X, \mathbb{R}) = \frac{\Gamma(X, C^\infty)}{d\Gamma(X, C^\infty)} \neq 0$$

In fact,

$$f \longmapsto \int_0^1 f(x)\,dx \in \mathbb{R} \text{ induces}$$

an iso

$$H^1(X, \mathbb{R}) \cong \mathbb{R}.$$

This is a special case of de Rham's thm.