1 De Rham's Theorem

Given an open subset $U \subset \mathbb{R}^n$, a C^∞-vector field is an $\mathbb{R}$-linear derivation

$$D : C^\infty(U) \to C^\infty(U)$$

This means that D satisfies the Leibnitz rule

$$D(fg) = fDg + gDf$$

The set of vector fields forms a $C^\infty(U)$-module $\mathrm{Vect}(U)$.

Lemma $\mathrm{Vect}(U)$ is a free module of rank n, with basis $\frac{\partial}{\partial x_i}$, where $x_1, \dots x_n$ are coordinates.

Def The space of C^∞ 1-forms is

$$\mathcal{E}^1(U) = \mathrm{Hom}_{C^\infty(U)}(\mathrm{Vect}(U), C^\infty(U))$$

And the space of p-forms

$$\mathcal{E}^p(U) = \Lambda^p \mathcal{E}^1(U)$$

These are free modules with basis
$$dx_I = dx_{i_1} \wedge \dots \wedge dx_{i_p}$$
where $dx_i \in \Sigma^1(U)$ is the dual basis to $\frac{\partial}{\partial x_i}$

Def The exterior derivative is a linear map
$$d : \Sigma^p(U) \longrightarrow \Sigma^{p+1}(U)$$
determined by
$$d(f_I\, dx_I) = \sum_{j=1}^{n} \frac{\partial f_I}{\partial x_j} dx_j \wedge dx_I$$

Lemma $d^2 = d \circ d = 0$.

We'll just check it for $p=0$.
$$d^2 f = d\left(\sum_{i=1}^{n} \frac{\partial f}{\partial x_i} dx_i \right)$$
$$= \sum_j \sum_i \frac{\partial^2 f}{\partial x_j \partial x_i} dx_j \wedge dx_i$$
$$= \sum_{i<j} \frac{\partial^2 f}{\partial x_j \partial x_i} dx_j \wedge dx_i - \sum_{i>j} \frac{\partial^2 f}{\partial x_j \partial x_i} dx_j \wedge dx_i$$
$$= 0$$

Thm/Def For any manifold X there exist a sheaf of C^∞_X-modules Σ^p_X with a morphism $d : \Sigma^p_X \to \Sigma^{p+1}_X$ s.t. for any isomorphis $X \supset V \xrightarrow[\varphi]{\sim} U \subseteq \mathbb{R}^n$ φ induces

$$\begin{array}{ccc} \Sigma^p(V) & \xrightarrow{d} & \Sigma^{p+1}(V) \\ \wr\| & & \wr\| \\ \Sigma^p(U) & \xrightarrow{d} & \Sigma^{p+1}(U) \end{array}$$

Sketch Given $x \in X$ let $C^\infty_{X,x}$ be the stalk of C^∞_X at x. We can define the tangent space at x

$$T_x = \{ D : C^\infty_{X,x} \to \mathbb{R} \mid D \text{ is an } \mathbb{R}\text{-linear derivation} \}$$

A vector field $\mathrm{Vect}_X(U)$ is a collection $D_x \in T_x$, $x \in U$, s.t. $\forall f \in C^\infty(U)$ the function

$$x \in U \longmapsto D_x(f_x) \in \mathbb{R}$$

is C^∞.

Then $Vect_X$ defines a sheaf of C^∞_X-modules. Define Σ^1_X as the dual module

$$\Sigma^1_X(U) = \mathrm{Hom}_{C^\infty_U}(Vect_U, C^\infty_U)$$

and

$$\Sigma^p_X = \Lambda^p \Sigma^1_X := (U \mapsto \Lambda^p \Sigma^1(U))^+$$

For $d : C^\infty_X \longrightarrow \Sigma^1_X$ we define it by

$$(df)(D) = D(f)$$

One checks that there is a unique extension $d : \Sigma^p_X \to \Sigma^{p+1}_X$ s.t.

$$d(f\alpha) = df \wedge \alpha + f d\alpha$$

$\forall f \in C^\infty(U)$, $\alpha \in \Sigma^p(U)$.

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Def The de Rham cohomology of a manifold

$$H^p_{DR}(X) = \frac{\ker\left[d : \Sigma^p(X) \to \Sigma^{p+1}(X)\right]}{\operatorname{im}\left[d : \Sigma^{p-1}(X) \to \Sigma^p(X)\right]}$$

Here is a basic computation that can be found in any book on manifolds (e.g. Spivak's Calculus on Manifolds)

Theorem (Poincaré's lemma)

$$H^p_{DR}(\mathbb{R}^n) = 0 \quad \forall p$$

i.e. if $d\alpha = 0$, then $\exists \beta$ s.t. $\alpha = d\beta$.

This fails in general, e.g. for $X = S^1$

De Rham's Thm

$$H^p_{DR}(X, \mathbb{R}) \cong H^p(X, \mathbb{R}_X)$$

Rmk Usually this is stated using singular cohomology, which won't get into.

Def A sheaf $\mathcal{F}$ is called **acyclic** if $H^i(X, \mathcal{F}) = 0 \quad \forall i > 0$.

Ex Flasque sheaves are acyclic and so are soft sheaves on metric spaces are acyclic.

The key result needed is:

Thm (Acyclic resolution thm)
If $0 \to \mathcal{F} \to \mathcal{Q}^0 \to \mathcal{Q}^1 \to \cdots$ is an exact sequence with $\mathcal{Q}^i$ acyclic, then

$$H^i(X, \mathcal{F}) \cong \frac{\ker\left(H^0(X, \mathcal{Q}^i) \to H^0(X, \mathcal{Q}^{i+1})\right)}{\operatorname{im}\left(H^0(X, \mathcal{Q}^{i-1}) \to H^0(X, \mathcal{Q}^i)\right)}$$

Pf Set

$$\begin{cases} K^{-1} = \mathcal{F} \\ K^n = \ker\left(\mathcal{Q}^{n+1} \to \mathcal{Q}^{n+2}\right) & n \geq 0 \end{cases}$$

Then we have exact seq.

$$0 \to K^n \to \mathcal{Q}^n \to K^{n+1} \to 0$$

Since $\mathcal{a}^i$ is acyclic we have

$$H^0(\mathcal{a}^0) \to H^0(\mathcal{K}^0) \to H^1(\mathcal{F}) \to 0$$

$$0 \to H^0(\mathcal{K}^0) \to H^0(\mathcal{a}^1) \to H^0(\mathcal{K}^1)$$

$$0 \to H^0(\mathcal{K}^1) \to H^0(\mathcal{a}^2)$$

$$\Rightarrow H^1(\mathcal{F}) = \frac{H^0(\mathcal{K}^0)}{H^0(\mathcal{a}^0)}$$

$$= \frac{\ker[H^0(\mathcal{a}^1) \to H^0(\mathcal{a}^2)]}{\operatorname{im} H^0(\mathcal{a}^0)}$$

The other cases are similar //

pf of de Rham

By the Poincaré lemma

$$0 \to \mathbb{R}_x \to \mathcal{E}^0_x \xrightarrow{d} \mathcal{E}^1_x \xrightarrow{d}$$

is an acyclic resolution. Now we can apply the previous theorem. //

The acyclic resolution theorem has many applications beyond de Rham's theorem, such as

(1) The standard method for defining $H^i(X, \mathcal{F})$ is through injective resolutions. Injective sheaves (whatever they are) are flasque and therefore acyclic. It follows that the standard method and the one we're using coincide.

(2) Godement showed that every sheaf $\mathcal{F}$ has a canonical flasque resolution

$$0 \to \mathcal{F} \to G^0(\mathcal{F}) \to G^1(\mathcal{F}) \to \cdots$$

Basically $G^0(\mathcal{F}) = G(\mathcal{F})$

$G^1(\mathcal{F}) = G(C(\mathcal{F}))$

etc.

Godement defined cohomology using his resolution. The theorem shows we get the same groups.

2. Kähler differentials

We want to explain the notion of 1-form in commutative algebra / algebraic geometry. Let k be a field and R a k-algebra. In analogy with what we did for manifolds, we can view the space of vector fields on Spec R as the space of k-linear derivations

$$\mathrm{Vect}(\mathrm{Spec}\, R) = \mathrm{Der}_k(R, R)$$

So it might seem that we should define a 1-form on Spec R as an element of the dual. However, this is where things diverge. We want to characterize the space of 1-forms by a universal property. Given an R-module M let

$$\mathrm{Der}_k(R, M) = \{ D : R \to M \mid D \text{ } k\text{-linear}, \; D(r_1 r_2) = r_1 Dr_2 + r_2 Dr_1 \}$$

Thm/Def There exists an R-module $\Omega_{R/k}$, called the module of **Kähler differentials**, such that it possesses a universal derivation $d \in \mathrm{Der}_k(R, \Omega^1_{R/k})$. This means given $\delta \in \mathrm{Der}_k(R, M)$, there exists a unique commutative diagram.

$$\begin{tikzcd} R \arrow[r, "d"] \arrow[dr, "\delta"'] & \Omega_{R/k} \arrow[d, dotted] \\ & M \end{tikzcd}$$

Sketch

The most direct pf/construction is to take the quotient of the free module on the symbols $\{dr \mid r \in R\}$ modulo relations

$$\begin{cases} d(r_1 + r_2) = dr_1 + dr_2 \\ d(r_1 r_2) = r_1\, dr_2 + r_2\, dr_1 \\ d r = 0, \quad r \in k. \end{cases}$$

A different construction is given in Matsumura's Commutative Algebra/Ring theory.

Ex 1 Let $R = k[x_1, \dots x_n]$

$$\Omega_{R/k} = \bigoplus R\,dx_i \cong R^n$$

with $df = \sum \frac{\partial f}{\partial x_i} dx_i$

Ex 2 Let $R = k[x_1 \dots x_n]/(g)$

$$\Omega_{R/k} \cong R^n / R\left(\frac{\partial g}{\partial x_1}, \dots \frac{\partial g}{\partial x_n}\right)$$

Suppose $k = \bar{k}$. We say $X = V(g)$ is singular if $\exists\, a \in X$ s.t.

$$\frac{\partial g}{\partial x_i}(a) = 0 \quad \forall i$$

Otherwise X is nonsingular

We define the cotangent space at $a \in X$ by

$$T^*_{X,a} = \Omega_{R/k} \otimes_R R/m_a \cong k^n / \left\langle \frac{\partial g}{\partial x_1}(a), \dots \frac{\partial g}{\partial x_n}(a) \right\rangle$$

where m_a is the maximal ideal $I(a) \subset R$.

We see that

<u>Lemma</u> $a \in X$ is nonsingular $\Leftrightarrow$ $\dim T^*_{X,a}$
$= n-1$
$(= \dim R)$

Here $\dim R$ is Krull dimension
(See Atiyah–Macdonald)

We turn this into a more general definition

<u>Def</u> Let X be an affine algebraic variety of $k = \bar{k}$ of dimension n
($=$ dim of coordinate ring R)

Then a point in the classical sense ($=$ closed point) $a \in X$ is
nonsingular if $T^*_{X,a} = \Omega_{R/k} \otimes R/m_a$
has dim n.