1 Kähler differentials

A **local ring** R is a ring with a unique maximal ideal m. The quotient $k = R/m$ is called the residue field. Let us assume that we also have an inclusion, $k \subset R$ such that the composite

$$k \to R \longrightarrow R/m = k$$

is the identity. This doesn't always happen, but it holds in the following key example

Ex Let k be an alg. closed field, and let S be finitely generated k-algebra. If $p \in \operatorname{Spec} S$ and $R = S_p$ is the localization, then R is a local ring with max ideal $m = pR$ s.t. $k = R/m$.

Prop/Def Suppose that R is a noetherian local ring which contains its residue field k as above. Then there is an isomorphism of k-vector spaces

$$\mathrm{Hom}_R(\Omega_{R/k}, k) = \mathrm{Hom}_k(m/m^2, k)$$

This is called the (Zariski) tangent space.

Sketch We have

$$\mathrm{Hom}_R(\Omega_{R/k}, k) = \mathrm{Der}_R(R, k)$$

by the definition of $\Omega_{R/k}$. Given $D \in \mathrm{Der}(R, k)$, $D|_m : m \to k$ is linear, such that $D(fg) = fDg + gDf = 0$ if $f, g \in m$. Therefore $D|_m(m^2) = 0$. So it induces a k-linear map $\delta : m/m^2 \to k$. Conversely, given $\delta \in \mathrm{Hom}(m/m^2, k)$ define

$$D(f) = \delta(f - ev(f))$$

where $ev : R \to k$ is the canonical map. One checks $D \in \mathrm{Der}_R(R, k)$.

and we have a bijection

$$D \leftrightarrow \delta$$

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Def A noetherian local ring R is **regular** if $\dim m/m^2 = \dim R$.
(In general, $\dim m/m^2 \geq \dim R$; see Atiyah-Macdonald)

Thm Let S be the coordinate ring of an affine alg. variety X over $k = \bar{k}$. Let $a \in X$, and let $R = S_{m_a}$. Then a is nonsingular $\iff$ R is a regular local ring.

Pf This follows from what we've said once we observe that
$$\dim R = \dim S$$
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Def A module is called **projective** if it is a direct summand of a free module.

Given a module M over a domain with fraction field K, define $\operatorname{rank} M = \dim K \otimes_R M$

Thm Given an (irreducible) affine variety X with coordinate ring R, X is nonsingular i.e. all points are nonsingular, iff $\Omega_{R/k}$ is a projective module of rank $= \dim X$.

pf We'll only prove one direction

$\Omega_{R/k}$ projective $\Rightarrow$ X nonsingular.

Let $k(x)$ = field of fractions of R
= function field of X.

and let $n = \dim X$.

We'll need the following facts from comm. alg (see Matsumura)

$k(x) \otimes_R \Omega_{R/k} = \Omega_{k(x)/k}$

is a vector space n over $k(x)$.

Lemma If M is a f.g R-mod., then for any $m \in \operatorname{Max} R$,

$\dim_{R/m} R/m \otimes M \geq \operatorname{rk} M$

Pf of lemma Follows from Nakayama's lemma.

Suppose that

$$\Omega_{R/k} \oplus M = R^N$$

for some M and N.

Then

$$(k(x) \otimes \Omega) \oplus (k(x) \otimes M) = k(x)^N$$

$\Rightarrow \quad rk\, \Omega = N - rk\, M \quad (1)$

$\forall\, p \in X$

$$(R/m_p \otimes \Omega) \oplus (R/m_p \otimes M) = (R/m_p)^N$$

$\Rightarrow \quad \dim T_p^* = N - \dim_{R/m_p} R/m_p \otimes M \quad (2)$

But, by lemma

$$\dim R/m_p \otimes M \geq rk\, M. \quad (3)$$

& $\quad \dim R/m_p \otimes \Omega \geq rk\, \Omega \quad (4)$

Therefore using (1), (2), (3), (4)

$$\dim T_p^* = rk\, \Omega = n$$

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2 Schemes

Given a continuous map

$$f : X \to Y$$

and a presheaf $\mathcal{F}$ on X, the **direct image** $f_* \mathcal{F}$ is a presheaf on Y defined by

$$(f_* \mathcal{F})(U) = \mathcal{F}(f^{-1} U)$$

with restrictions defined via:

$$\begin{array}{ccc} (f_* \mathcal{F})(U) & \longrightarrow & (f_* \mathcal{F})(V) \\ \| & & \| \\ \mathcal{F}(f^{-1} U) & \longrightarrow & \mathcal{F}(f^{-1} V) \end{array}$$

Lemma If $\mathcal{F}$ is a sheaf, then $f_* \mathcal{F}$ is also a sheaf.

pf exercise!

There is an operation from (pre)sheaves on Y to (pre)sheaves on X call the **inverse image**, characterised by a natural iso:

$$\mathrm{Hom}_{Ab(X)}(f^{-1}\mathcal{F}, \mathcal{G}) \cong \mathrm{Hom}(\mathcal{F}, f_*\mathcal{G})$$

In other words f^{-1} is left adjoint to f_* (N.B. It's usually not the inverse to f_*.) The only thing we note is that

$$(f^{-1}\mathcal{F})_p \cong \mathcal{F}_{f(p)}$$

Further details can be found in Hartshorne.

Recall that ringed space is a space X together with a sheaf R of commutative rings. We make these into a category

Def A morphism $(f, f_\#): (X, R) \to (Y, S)$ of ringed spaces is a continuous map $f: X \to Y$ plus a morphism $f_\#: S \to f_* R$ of sheaves of rings (equivalently we could specify the adjoint $f^\#: f^{-1}S \to R$.)

Ex 1 C^∞ (resp. regular) maps between manifolds (resp. alg. varieties) are morphism in this sense, where $f_\#(g) = g \circ f$

Ex 2 if $h : R \to S$ is a homorphism of rings, then we get an an induced morphism $(f, f_\#) : \text{Spec } S \to \text{Spec } R$

$$f : \text{Spec } S \to \text{Spec } R, \quad f(p) = h^{-1} p.$$

is continuous. $f_\#$ is given by

$$f_\# \quad \mathcal{O}_{\text{Spec} R}(D(g)) \to \mathcal{O}_{\text{Spec} R}(D(h(g)))$$

$$\begin{array}{ccc} \| & & \| \\ R[g^{-1}] & \xrightarrow{f} & S[h(g)^{-1}] \end{array}$$

An obvious question is whether every morphism of ringed spaces $\text{Spec } S \to \text{Spec } R$ comes from a ring homorphism. The answer is no, see Hartshorne p 74 for a counterexample.

If we refine the definition, we will get a positive answer.

Def A ringed space $(X, \mathcal{R})$ is called a **locally ringed space** if $\forall x \in X$, $\mathcal{R}_x$ is a local ring i.e. it has a unique maximal ideal m_x. These are made into a category with morphisms

$$(f, f^{\#}) : (X, \mathcal{R}) \to (Y, \mathcal{S})$$

of ringed spaces s.t. $\forall x \in X$

$$f^{\#}(m_x) \subseteq m_{f(x)}$$

(such a homomorphism of local rings is called a local homomorphism)

Ex 1 A C^∞-manifold is a locally ringed space. A C^∞ map induces a morphism of locally ringed spaces.

Ex 2 An affine scheme is locally ringed. A homomorphism $R \to S$ induces a morphism of locally ringed spaces $\operatorname{Spec} S \to \operatorname{Spec} R$.

Thm

$$\mathrm{Hom}_{\text{rings}}(R, S) \cong \mathrm{Hom}_{\text{loc. ring sp}}(\mathrm{Spec}\, S, \mathrm{Spec}\, R)$$

(See Hartshorne p 73)

Def A **scheme** is a locally ringed space $(X, \mathcal{O}_X)$ s.t. every pt has an open nbhd U s.t. $(U, \mathcal{O}_X|_U)$ is isomorphic to an affine scheme as a locally ringed space.

Of course an affine scheme is an example. Here is a new example

Ex Let R be a ring. The diagram

$$R[x] \rightarrow R[x, x^{-1}] \leftarrow R[x^{-1}]$$

gives a diagram

$$\operatorname{spec} R[x, x^{-1}] = U_{01}$$

$$U_0 = \operatorname{spec} R[x] \hookleftarrow \qquad \hookrightarrow U_1 = \operatorname{Spec} R[x^{-1}]$$

$$\text{i.e. } U_0 \cong \mathbb{A}^1_R \qquad U_1 \cong \mathbb{A}^1_R$$

$\mathbb{P}^1_R$ is obtained by gluing U_0 to U_1 along U_{01}

(See Hartshorne p 75 for details)

This is not affine: when $R = k$ is a field we saw $H^0(\mathbb{P}^1_k, \mathcal{O}_{\mathbb{P}^1}) = k$.

If it were affine then we would have $\mathbb{P}^1_k \cong \operatorname{spec} k$, but is false.