Clifford algebras, Fermions and Spin chains

Birgit Wehefritz-Kaufmann

Physics Department, University of Connecticut, U-3046, 2152 Hillside Road, Storrs, CT 06269-3046

Abstract. We show how using Clifford algebras and their representations can greatly simplify the analysis of integrable systems. In particular, we apply this approach to the XX-model with non-diagonal boundaries which is among others related to growing and fluctuating interfaces and stochastic reaction-diffusion systems. Using this approach, we can not only diagonalize the system, but also find new hidden symmetries.

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1 Introduction

There are three complex vector spaces which happen to have the same dimension 2^L (1) $V_1 = (\mathbf{C}^2)^{\otimes L}$ (2) $V_2 = \Lambda^* \mathbf{C}^L$ and (3) V_3, the unique irreducible representation of the complex Clifford algebra Cl_{2L} (see e.g. [1] for the last statement). These three vector spaces are basically studied by two groups of people, the ones working in integrable systems and the ones working on math/physics related to group theory. Each group has their own intrinsic reasons and interpretation for

each of these spaces. Our aim is to bring these two groups of people together by providing a general framework linking these spaces and interpretations together. We apply this philosophy to a concrete problem of the boundary XX–chain and present new results that can be obtained by systematically following this connection which we first explicate.

2 Clifford algebras and Spin Systems: A digest

The shortest way to define the Clifford algebra for a given $\mathbf{R}$–vector space W of dimension n with a fixed quadratic form Q is to give a presentation in terms of generators and relations. Let e_i be a basis of W and let $\langle\ ,\ \rangle$ be the bilinear form associated to Q. Then $C(W,Q) := \langle e_i : \{e_i,e_j\} = 2\langle e_i,e_j\rangle\ \rangle$[1]; in particular $C_k := C(\mathbf{R}^k,-\mathbb{1}_k)$ and $Cl_k := \mathbf{C}\otimes_{\mathbf{R}} C_k$. Here $\{e_i,e_j\} = e_ie_j - e_je_i$ and $\mathbb{1}_k$ is the $k\times k$ identity matrix. Notice that for all non–degenerate Q, $\mathbf{C}\otimes_{\mathbf{R}} C(\mathbf{R}^k,Q) \simeq Cl_k$.

We will first focus on two "standard" quadratic forms on $\mathbf{C}^{2L}$. The first is $Q = \mathbb{1}_{2L} = diag(1,\ldots,1)$ where we used the basis $c_1^+,\ldots,c_L^+, c_1^-,\ldots,c_L^-$. The algebra Cl_{2L} is then presented as

$$\{c_m^\mu,c_n^\nu\} = 2\delta_{m,n}^{\mu,\nu} \tag{1}$$

The generators in this presentation are usually called "Clifford Operators". The second form is $Q = \omega_{2L} = \frac{1}{2}\begin{pmatrix} 0 & \mathbb{1}_L \\ \mathbb{1}_L & 0\end{pmatrix}$ where we used the basis $b_1,\ldots,b_L, a_1,\ldots,a_L$. This leads to the "Fermion" representation

$$\{b_n,a_m\} = \delta_{n,m},\quad \{b_n,b_m\} = 0,\quad \{a_n,a_m\} = 0 \tag{2}$$

These two presentations of the same algebra are related by

$$b_n = \frac{1}{2}(c_n^+ + ic_n^-), a_n = \frac{1}{2}(c_n^+ - ic_n^-);\quad c_n^+ = b_n + a_n, c_n^- = \frac{1}{i}(b_n - a_n)$$

A (spin 1/2) spin–chain is an operator $H \in End((\mathbf{C}^2)^{\otimes L})$. Notice that due to the general properties of Clifford algebras [1, 2] $Cl_{2L} \simeq$

[1] We will usually not distinguish in notation between elements of W and elements in the Clifford algebra. If this becomes necessary, we denote the inclusion of W into $C(W,Q)$ by ι.

$Cl_2^{\hat{\otimes} L} \simeq M_2(\mathbf{C})^{\hat{\otimes} L}$. Here $\hat{\otimes}$ is the tensor product for $\mathbf{Z}/2\mathbf{Z}$–graded algebras. So, although $Cl_2 \simeq End(\mathbf{C}^2) = M_2(\mathbf{C})$ the Clifford algebra Cl_L does not readily act on $V = (\mathbf{C}^2)^{\otimes L}$ due to the inequality of graded $\hat{\otimes}$ and non–graded $\otimes$ tensor products. The remedy is the Jordan–Wigner transformation (JWT) [3] which is "highly non–local".

$$Cl_{2L} \to Cl_2^{\otimes L}; \quad \tau_j^{+,-} \mapsto (\prod_{i=1}^{j-1} \sigma_i^z)\sigma_j^{x,y} \tag{3}$$

Here $\sigma^x, \sigma^y, \sigma^z$ are the Pauli–Spin–matrices and we used the notation $\sigma_j := \bigotimes_{k=1}^{j-1} \mathbb{1}_2 \otimes \sigma \otimes \bigotimes_{k=j+1}^{L} \mathbb{1}_2$; we also set $\sigma^{\pm} = \frac{1}{2}(\sigma^x \pm i\sigma^y)$.

3 Fermionization of Spin chains

With the help of the JWT one can write many spin–chain problems in terms of Fermions [4, 5] (see also [6, 7] for more examples). The main point is that simple quadratic forms in the non–local operators τ

$$H = \sum_{n=1}^{L} \Lambda_n i\tau_n^- \tau_n^+ = \sum_{n=1}^{L} 2\Lambda_n b_n a_n - \sum_{n=1}^{L} \Lambda_n = \sum_{n=1}^{L} 2\Lambda_n N_n + E_0 \tag{4}$$

give rise to physically interesting operators which are naturally written in terms of the σ matrices and vice–versa. The problem of Fermionization is to determine if a given Hamiltonian can be brought into the form $H = \frac{1}{2}\sum' M_{m,n}^{\mu,\nu} \tau_m^\mu \tau_n^\nu$ and then diagonalized, i.e. brought into the form (4). Here $\sum'$ is the sum over all $(m,\mu) \neq (n,\nu)$. In this expression H is a *symmetric form* on the *odd* vector space $\iota(\mathbf{C}^{2L})$. This problem is equivalent to "diagonalizing" the anti–symmetric form M on the *even* vector space $\mathbf{C}^{2L}$. This is done by a change of base $\mathbf{\Psi}$. With $\mathbf{\Lambda} = diag(\Lambda_1, \ldots, \Lambda_L)$ the resulting matrix equation reads $\mathbf{\Psi}^t M \mathbf{\Psi} = \begin{pmatrix} 0 & i\mathbf{\Lambda} \\ -i\,\mathbf{\Lambda} & 0 \end{pmatrix}$. In order to preserve the Clifford presentation (1), we also need that $\mathbf{\Psi}^t \mathbf{\Psi} = \mathbb{1}_{2L}$. Notice if we work with Fermions instead then a base change $\mathbf{\Psi}$ would have to satisfy $\mathbf{\Phi}^t \omega_{2L} \mathbf{\Phi} = \omega_{2L}$.

4 The XX-chain with boundaries and a new conserved quantity

By enlarging the vector space to $\mathbf{C}^2 \otimes \mathbf{C}^{\otimes L} \otimes \mathbf{C}^2$ [8] the rationale of the sections above can be applied to the XX–chain with boundaries[9]

$$H = \frac{1}{2}\sum_{j=1}^{L-1}\left[\sigma_j^+\sigma_{j+1}^- + \sigma_j^-\sigma_{j+1}^+\right] + \frac{1}{\sqrt{8}}\left[\alpha_-\sigma_1^- + \alpha_+\sigma_1^+ + \alpha_z\sigma_1^z + \beta_+\sigma_L^+ + \beta_-\sigma_L^- + \beta_z\sigma_L^z\right]$$

The projection to the original problem [9] is non–trivial and its understanding in terms of Clifford representations is still an interesting open problem; in particular, since it seems that complex Clifford algebras of odd dimensional vector spaces make an appearance [10].

Having obtained this we can "pull–back" operators that commute with H in the form (4) to get new conserved quantities. One such operator is total Fermion number operator $\mathscr{F}_{\rm tot} = \sum_{n=1}^{L+1} b_n a_n$ (for another such operator see [11]). We include the expression in terms of the σ matrices [12] to show the power of the transformation.

$$\begin{aligned}
\mathscr{F}_{\rm tot} = {} & \frac{1}{4L+4}\sum_{\substack{j,k=1\\ j+k\ \text{odd}, j<k}}^{L}(-1)^{\frac{j+k+1}{2}}(\sigma_{j+1}^z\ldots\sigma_{k-1}^z) \\
& \left[\left(\frac{\sin(\frac{\chi(j-k)}{L+1})}{\sin(\frac{\pi(j-k)}{2L+2})} + \frac{\sin(\frac{\chi(j+k)}{L+1})}{\sin(\frac{\pi(j+k)}{2L+2})}\right)(\sigma_j^x\sigma_k^y - \sigma_j^y\sigma_k^y) + \left(\frac{\cos(\frac{\chi(j+k)}{L+1})}{\sin(\frac{\pi(j+k)}{2L+2})}\right)(\sigma_j^y\sigma_k^y - \sigma_j^x\sigma_k^x)\right] \\
& + \frac{1}{\sqrt{8}(L+1)}\left\{\sum_{\substack{k=1\\ k\ \text{odd}}}^{L-1}(-1)^{\frac{k+1}{2}}\left[\frac{\cos(\frac{\chi k}{L+1})}{\sin(\frac{\pi k}{2L+2})}\sigma_1^z\sigma_2^z\ldots\sigma_{k-1}^z\sigma_k^x\right]\right. \\
& \left. + \sum_{\substack{k=2\\ k\text{even}}}^{L}\left[(-1)^{\frac{L+k+2}{2}}\frac{\cos(\frac{\chi k}{L+1})}{\cos(\frac{\pi k}{2L+2})}\sigma_k^x + (-1)^{\frac{k+L}{2}}\frac{\sin(\frac{\chi k}{L+1})}{\cos(\frac{\pi k}{2L+2})}\sigma_k^y\right]\left(\sigma_{k+1}^z\ldots\sigma_{L-1}^z\sigma_L^z\right)\right\}
\end{aligned} \tag{5}$$

Since we have super-selection rules for the state space, we can look at the partition function $\mathscr{Z}_m$ of H restricted to the sector with m Fermions. In the continuum limit [13], we obtain (see [12] for further details)

$$\mathscr{Z}_m = \lim_{L\to\infty}\left(\mathrm{tr}\, z^{\frac{L}{\pi}\sum_{n=0}^{L} 2\lambda_n N_n}\right) = z^{\frac{m\chi}{\pi}+\frac{m}{2}}\frac{z^{\frac{m(m+1)}{2}} + (1-z^m)z^{\frac{(m-1)m}{2}}}{(1-z)(1-z^2)\ldots(1-z^m)} \tag{6}$$

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