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Clifford representations in integrable systems

Birgit Wehefritz-Kaufmann[a)]
Physics Department, University of Connecticut, U-3046, 2152 Hillside Road, Storrs, Connecticut 06269-3046

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In this paper we analyze integrable systems from a Clifford algebra point of view. This approach allows us to give a clear representation theoretic exposition of techniques used in spin systems, thereby showing their naturality. We then extend this approach to the analysis of the XX-model with nondiagonal boundaries which is among others related to growing and fluctuating interfaces and stochastic reaction-diffusion systems. With this rationale, it is possible to diagonalize the system and find new hidden conservation laws. © *2006 American Institute of Physics.*
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INTRODUCTION

The use of Clifford operators to "fermionize" a problem goes back to Ref. 1. The potential this approach has in solving spin-chain problems was first demonstrated in Ref. 2; see Refs. 3–6 for other early sources. There has been a lot of work in this direction, Refs. 7–11 to name a few. We will first give a novel presentation of this "classic" connection between spin-chain Hamiltonians and their fermionization using Clifford operators. This is done in a concise mathematical way focusing on the Clifford algebra aspects. This allows us to explain properties, procedures, and characteristics which appear complex and complicated in the spin-chain picture in a clean straightforward fashion as direct consequences of the mathematical setup. We hope that this treatment may help to bring the two communities, the people working in mathematical physics using group theory and the people working on integrable systems, closer together. We then show that going beyond the classical quadratic Hamiltonians, we can in certain cases, namely the XX-model, extend to include boundary terms. There are some subtleties here. First we have to actually enlarge the chain to be able to obtain a quadratic Hamiltonian, one of whose sectors is the original problem. Then, after the fermionization, we have to project to the smaller problem, which is nontrivial in the fermion language. As we explain, the actual calculation of the fermionization is a matrix valued problem due to the theorem[1,12,13] about the uniqueness of the irreducible Clifford module of the complex Clifford algebra based on an even dimensional vector space. We then go on to analyze operators which commute with the XX-chain Hamiltonian with boundary terms. These operators can be shown to commute, but due to the nonlocal nature of the Jordan-Wigner transformation *and* the projection, they become highly nontrivial in the original spin-chain picture. Lastly, we comment on how this operator behaves in the thermodynamic limit.

I. CLIFFORD ALGEBRAS AND SPIN CHAINS: A DIGEST

A. The Clifford algebra and fermions

A Clifford algebra $C(W,Q)$ is a universal algebra associated with a given $\mathbb{R}$-vector space W with a quadratic form Q. The universal property is that any linear map $j: W \to A$ of W to an associative $\mathbb{R}$-algebra A with unit 1 which satisfies $j(w)^2 = Q(w)1$ factors through $C(W,Q)$ uniquely up to isomorphism. To be really careful of course the universal object $C(W,Q)$ comes with a map $\iota: W \to C(W,Q)$ and j factors through ι. For W with $\dim_{\mathbb{R}} W = k$ and a basis (e_i) of W,

[a)]Electronic mail: kaufmann@phys.uconn.edu the algebra $C(W,Q)$ is a quadratic algebra generated by $e_i : i=1,\ldots,k$ and the relations $\forall n,m:\{e_n,e_m\}=2\langle e_n,e_m\rangle$ where $\langle\cdot,\cdot\rangle$ is the bilinear form associated with the quadratic form Q. (Technically, for this one has to assume that one is not in characteristic 2. This is fine since we will be working over $\mathbb{R}$ and $\mathbb{C}$ that is in characteristic 0.) We will usually not distinguish v and $\iota(v)$ in our notation, but sometimes it will be necessary. It can easily be seen that $\dim_{\mathbb{R}}(C(W,Q))=2^n$ for W as above and any Q. See, e.g., Refs. 12 and 13 for these results.

The classical example is the Clifford algebra $C_k := C(\mathbb{R}^k, \mathrm{diag}(-1,\ldots,-1))$. This algebra can be written as generated by e_n subject to the conditions $e_n^2=-1, e_m e_n + e_n e_m = 0$. Now there are the standard isomorphisms $C_2 \simeq \mathbb{H} \simeq \mathrm{su}(2)$. Where one sends $e_1 \mapsto I, e_2 \mapsto J$, here I, J, and K are the usual quaternions and in the last step one represents I and J by the usual 2×2 matrices. Alternatively one can of course use the first two elements of any cyclic permutation of the matrices representing I, J, and K.

We will consider the complexified Clifford algebra $Cl_k := \mathbb{C}\otimes_{\mathbb{R}} C_k$. Notice that over the complexes all nondegenerate forms are conjugate and we will only work with such forms. Thus the Clifford algebras over $\mathbb{C}$ do not depend on the particular form of Q as long as it is nondegenerate. In general it can even be shown that $Cl_{2L} \simeq M_{2^L}(\mathbb{C})$, the full matrix algebra, see, e.g., Refs. 12 and 13. To pass from one quadratic form to another one makes a base change on the underlying vector space. This gives a change in presentation of the algebra. The algebra Cl_2 for the quadratic form $\mathrm{diag}(1,1)$, for instance, is just $M_2(\mathbb{C})$, and we can represent it via $e_1' \mapsto \sigma_x, e_2' \mapsto \sigma_y$. Since $\sigma x = -iK$ *and* $\sigma_y = -iJ$ *in their standard matrix representations (see, e.g., Ref. 13), we obtain an isomorphism with* $Cl_2 = \mathbb{C}\otimes_{\mathbb{R}} C_2$ *by using the permutation as mentioned above to represent the Clifford algebra and then use the complex base change* $e_j' = (-i)e_j$. *Of course* $JK=I$ *and* $\sigma_x\sigma_y = -KJ = I = i\sigma_z$.

In general, there are two standard quadratic forms: the first is given by the $k\times k$ matrix $\mathbb{1}_k = \mathrm{diag}(1,\ldots,1)$ and the second one which exists on $\mathbb{C}^{2L}$ is $\omega_{2L} = \frac{1}{2}\begin{pmatrix} 0 & \mathbb{1}_L \\ \mathbb{1}_L & 0 \end{pmatrix}$. The factor $\frac{1}{2}$ is added to cancel the factor of 2 in the relations. In the case that we are in even dimension that is in $\mathbb{R}^{2L}$ or after complexification in $\mathbb{C}^{2L}$, we fix the following notation for the basis elements. In the first case we denote the basis vectors $c_1^+,\ldots,c_L^+,c_1^-,\ldots,c_L^-$, and in the second case we will enumerate basis vectors $b_1,\ldots,b_L,a_1,\ldots,a_L$. This means that in the case of $\mathbb{C}^{2L}$ for the first basis we get the relations

$$\{c_m^\mu, c_n^\nu\} = 2\delta_{m,n}^{\mu,\nu} \tag{1}$$

for the generators of the Clifford algebra Cl_{2L} corresponding to this basis. These operators are usually called Clifford operators. For the second basis of $\mathbb{C}^{2L}$ we obtain the following relations for the generators of the Clifford algebra Cl_{2L}

$$\{b_n,a_m\} = \delta_{n,m}, \quad \{b_n,b_m\} = 0, \quad \{a_n,a_m\} = 0. \tag{2}$$

These operators are usually called fermion operators. (In a matrix representation one frequently also postulates $a_n^\dagger = b_n$. We will not impose this at the moment.) One can think of the b_n as creation and the a_n as annihilation operators of the nth fermion. The operator $N_n = b_n a_n$ then has eigenvalues of 0 or 1 corresponding to whether the fermion is present or not. The fermions can also just be seen as a representation of Cl_{2L} on the exterior algebra $\Lambda^*\mathbb{C}^L$. In the presentation of Eq. (2) we can write the Clifford algebra as $Cl^- \oplus Cl^0 \oplus Cl^+$, where Cl^- is the subalgebra generated by the a_n, Cl^+ is the one generated by the b_n, and Cl^0 is the center generated by $\mathbb{C}$. The Fock space representation R_{Fock} is then given as follows; let $R_{|vac\rangle}$ be the one-dimensional representation of $Cl^0 \oplus Cl^-$ on $\mathbb{C} = \mathbb{C}|vac\rangle$ for which $1|vac\rangle = |vac\rangle$ and $Cl^-|vac\rangle = 0$ then $R_{\mathrm{Fock}} = Cl \otimes_{Cl^0\oplus Cl^-} R_{|vac\rangle}$. Here $R_{\mathrm{Fock}} \simeq \Lambda^*\mathbb{C}^L$ as vector spaces and the Cl module structure is given by left multiplication.

The two sets of generators of Cl_{2L} are related by the simple base change on $W = \mathbb{C}^{2L}$,

$$b_n = \frac{1}{2}(c_n^+ + ic_n^-), \quad a_n = \frac{1}{2}(c_n^+ - ic_n^-),$$

$$c_n^+ = b_n + a_n, \quad c_n^- = \frac{1}{i}(b_n - a_n). \tag{3}$$

In the Clifford operator basis the role of the operator N_n is played by $ic_n^- c_n^+$ which has eigenvalues ± 1.

B. Spin chains

A spin chain of length L is a $\mathbb{C}$-vector space V which is a tensor product of L copies of a su(2) representation. A spin-chain Hamiltonian is an operator H which acts on such a V. (Sometimes a spin chain is taken to mean V together with H.) We will only be concerned with spin $\frac{1}{2}$. That is, $V=\otimes_{i=1}^{L}\mathbb{C}^2$ and $H:V\to V$ which means that $H\in GL(2^L,\mathbb{C})$. (Although the notation H is suggestive of both Hermitian and Hamiltonian, we will not restrict to the case that H is Hermitian.) The copies of $\mathbb{C}^2$ are usually called sites. Many of the interesting Hamiltonians are obtained by using linear and quadratic expressions in the Pauli-spin matrices σ_x, σ_y, and σ_z. It is well known that the Pauli matrices together with the identity matrix with $\mathbb{C}$ coefficients form a basis of $M_2(\mathbb{C})\simeq\mathbb{C}\otimes\mathbb{H}\simeq\mathbb{C}^4$ as vector spaces.

Examples of such spin chains are the XX, XY, and XYZ or Heisenberg model. In particular, the XX chain has the Hamiltonian:

$$H_{\mathrm{XX}} = \frac{1}{2}\sum_{j=1}^{L}[\sigma_j^+\sigma_{j+1}^- + \sigma_j^-\sigma_{j+1}^+]. \tag{4}$$

Here we adopted the usual notation $\sigma_j := \mathbb{1}_2\otimes\ldots\otimes\mathbb{1}_2\otimes\overset{\overset{j}{\downarrow}}{\sigma}\otimes\mathbb{1}_2\otimes\ldots\otimes\mathbb{1}_2$, where σ is inserted at the jth spot.

C. The Jordan-Wigner transformation

We recall that all the Clifford algebras are $\mathbb{Z}/2\mathbb{Z}$ graded. This can either be seen by using the involution on $C(W,Q)$ generated by $\iota(w)\mapsto-\iota(w)$ for $w\in W$ or the fact that the algebra is quadratic and hence the $\mathbb{Z}$ grading of the tensor algebra descends to a $\mathbb{Z}/2\mathbb{Z}$ grading on $C(W,Q)$. This splits the Clifford algebra into its even and odd parts, $C(W,Q)=C(W,Q)^{\mathrm{even}}\oplus C(W,Q)^{\mathrm{odd}}$.

A wonderful fact about Clifford algebras is that they satisfy $C(W\oplus W',P\oplus Q)\simeq C(W,P)\hat{\otimes}C(W',Q)$. Here it is important that we used $\hat{\otimes}$, that is, the tensor product as $\mathbb{Z}/2\mathbb{Z}$ graded spaces. Using this property, we see that $Cl_{2L}\simeq Cl_2^{\hat{\otimes}L}\simeq M_2(\mathbb{C})^{\hat{\otimes}L}$. But we should be careful that $M_2(\mathbb{C})^{\hat{\otimes}L}\neq M_2(\mathbb{C})^{\otimes L}$. So we cannot directly identify the spin-chain operators with Clifford operators, since the former are nongraded tensor products while the latter are graded. This is easily seen since σ_i commutes with σ_j for $i\neq j$ in the spin-chain case, while they should anticommute if they would be Clifford operators. Indeed this is forced by considering $\hat{\otimes}$.

This obstacle was overcome by Ref. 1 (see Ref. 2) by the following isomorphism of $Cl_{2L}\to Cl_2^{\otimes L}$:

$$\tau_j^{+,-} = \left(\prod_{i=1}^{j-1}\sigma_i^z\right)\sigma_j^{x,y}. \tag{5}$$

It can easily be checked that these operators satisfy Eq. (1). As noticed in Ref. 1, see also Ref. 13, this is the only irreducible module of Cl_{2L}—up to isomorphism of course. This can most quickly be seen by using the fact that Cl_{2L} is a matrix algebra over $\mathbb{C}$ and hence Morita equivalent to $\mathbb{C}$.

D. Free fermions and fermionization

Comparing the two paragraphs, we see that the Clifford operators, that is elements of Cl_{2L} after using the Jordan-Wigner transformation, are operators on the spin-chain vector space V.

Moreover, using the operators defined in Eq. (5), we have Clifford operators on the spin chain and using the transformation (3), we obtain fermion operators. This allows us to write down "simple" Hamiltonians,

$$H = \sum_{n=1}^{L} \Lambda_n i \tau_n^- \tau_n^+, \tag{6}$$

or in terms of the associated fermion operators:

$$H = \sum_{n=1}^{L} 2\Lambda_n b_n a_n - \sum_{n=1}^{L} \Lambda_n = \sum_{n=1}^{L} 2\Lambda_n N_n + E_0. \tag{7}$$

In this form it is clear that H is the Hamiltonian of L free fermions whose energies are Λ_n. The second summand is the constant term corresponding to the Fermi sea.

Now as mentioned the representation (5) is unique up to isomorphism, so we could also first apply a base change to the operators $\tau_{m,n}^{\mu,\nu}$ to obtain operators $T_{m,n}^{\mu,\nu}$. Since the τ are images of the basis elements of W, this amounts to using a change of basis $\mathbf{\Psi}$ on the underlying vector space W of the Clifford algebra. Notice that W is different from $V=(\mathbb{C}^2)^{\otimes L}$ and that $\dim W=2L$, so that $\mathbf{\Psi}$ is a $2L\times 2L$ matrix. If we want that the operators T are still Clifford operators, that is they obey Eq. (1), then $\mathbf{\Psi}$ has to be an orthogonal transformation, that is, $\mathbf{\Psi}^t\mathbf{\Psi}=\mathbb{1}_{2L}$. Of course, in the fermionic case (7), the change of basis transformations $\mathbf{\Phi}$ preserving the relations (2) is the one that preserves ω: $\mathbf{\Psi}^t\omega_{2L}\mathbf{\Psi}=\omega_{2L}$.

Using these transformations, we obtain a class of Hamiltonians of the form

$$H = \sum_{n=1}^{L} \Lambda_n i T_n^- T_n^+. \tag{8}$$

Unravelling the definitions, we thus obtain spin chains describing free fermions. As spin-chain Hamiltonians, that is, expressed in terms of the σ-matrices, the form of H is highly nontrivial due to the nonlocal nature of the Jordan-Wigner transformation.

The fermionization problem is the inverse problem: When can I write a given spin chain in terms of free fermions? There is a way to go about this in case the Hamiltonian is a quadratic form in the elements τ,

$$H = \frac{1}{2} \sum_{\substack{(m,\mu),(n,\nu) \\ \text{s.t.}(m,\mu)\neq(n,\nu)}} M_{m,n}^{\mu,\nu} \tau_m^{\mu} \tau_n^{\nu}. \tag{9}$$

This is, for instance, the case if H is quadratic in the $\sigma^{\pm}$, $H=\Sigma A_{m,n}^{\mu,\nu}\sigma_m^{\mu}\sigma_n^{\nu}$, and one has only nearest neighbor interactions: $A_{m,n}^{\mu,\nu}=0$ if $|m-n|\neq 1$. Although we postulated that there are no diagonal entries, these entries would not pose any real problems, since due to the equation $(\tau_m^{\mu})^2=1$ they would just contribute a constant term.

In this case, we can think of M as a quadratic form on the image $\iota(W)$ of W. Now due to the grading $\iota(W)$ is an *odd* vector space, so that the matrix M for a "symmetric" quadratic form will be skew symmetric. The aim now is to find a transformation of basis on W which makes H diagonal in the Clifford basis corresponding to the new basis as in Eq. (8). Let $\mathbf{\Lambda}=\mathrm{diag}(\Lambda_1,\ldots,\Lambda_L)$, then the matrix problem one has to solve is $\mathbf{\Psi}^t M\mathbf{\Psi}=\begin{pmatrix} 0 & i\mathbf{\Lambda} \\ -i\mathbf{\Lambda} & 0 \end{pmatrix}$. Recalling that in order to preserve the fermion presentation of the Clifford algebra $\mathbf{\Psi}^t\mathbf{\Psi}=\mathbb{1}$, this reads

$$M\mathbf{\Psi} = \mathbf{\Psi}\begin{pmatrix} 0 & i\mathbf{\Lambda} \\ -i\mathbf{\Lambda} & 0 \end{pmatrix}. \tag{10}$$

This is the type of equation that is usually obtained by calculating the commutators of the operators T with the Hamiltonian;[2,7–11] here we find it by purely Clifford algebra considerations. Transforming to the equations by writing $\mathbf{\Psi}=(\mathbf{\Psi}^{+}\mathbf{\Psi}^{-})$ and $\mathbf{\Phi}^{+}=\mathbf{\Psi}^{+}+i\mathbf{\Psi}^{-},\mathbf{\Phi}^{-}=\mathbf{\Psi}^{+}-i\mathbf{\Psi}^{-}$, the equations become the eigenvalue problem:

$$M\mathbf{\Phi}^{\pm} = \pm\mathbf{\Phi}^{\pm}\mathbf{\Lambda}. \tag{11}$$

Since the matrix $\mathbf{\Psi}$ is a base change on W which was of dimension $2L$, we get $2L$ eigenvalues. These come in pairs of $\pm\Lambda_n$. This is clear from the skew symmetry of M. Moreover looking at Eq. (8), we see that the change from Λ_n to $-\Lambda_n$ does not affect the spectrum of H. This symmetry corresponds to sending one of the $T_n \to -T_n$.

II. THE XX-MODEL WITH BOUNDARIES

Now the above procedure works well for Hamiltonians such as the Ising, XX, XY, Heisenberg-Ising chains,[2,15] and other models.

With some work one can also sometimes include boundary terms. For this we consider the Hamiltonian of Ref. 14 specialized to

$$H=\frac{1}{2}\sum_{j=1}^{L-1}[\sigma_j^{+}\sigma_{j+1}^{-}+\sigma_j^{-}\sigma_{j+1}^{+}]+\frac{1}{\sqrt{8}}[\sigma_1^{-}+\sigma_1^{+}+\exp(i\chi)\sigma_L^{+}+\exp(-i\chi)\sigma_L^{-}]. \tag{12}$$

A. Extending the chain

The H of Eq. (12) is not quadratic. On the other hand we can add the sites 0 and $L+1$ as in Refs. 16 and 17 and consider the Hamiltonian

$$H_{\text{long}}=\frac{1}{2}\sum_{j=1}^{L-1}[\sigma_j^{+}\sigma_{j+1}^{-}+\sigma_j^{-}\sigma_{j+1}^{+}]+\frac{1}{\sqrt{8}}[\sigma_0^{x}\sigma_1^{-}+\sigma_0^{x}\sigma_1^{+}+\exp(i\chi)\sigma_L^{+}\sigma_{L+1}^{x}+\exp(-i\chi)\sigma_L^{-}\sigma_{L+1}^{x}], \tag{13}$$

which acts on $V_{\text{long}}=\otimes_{i=0}^{L+1}\mathbb{C}^2$. We see that H_{long} is of the form (9), since it is quadratic in the σ and has only nearest neighbor interactions. Thus we can apply the procedure of Sec. I D and express H_{long} in terms of $L+2$ fermions. From the form of H_{long} we see that the matrices σ_0^x and σ_{L+1}^x commute with H_{long}, so the spectrum of H_{long} decomposes into four sectors corresponding to the eigenvalues ± 1 of σ_0^x and σ_{L+1}^x. That is, we can write $V_{\text{long}}=\oplus_{(\mu,\nu)\in(\mathbb{Z}/2\mathbb{Z})^2}V_{\mu\nu}$. Here the notation $V_{\mu\nu}$ naturally means that $\sigma_0^x|V_{\mu\nu}=\mu id$ and $\sigma_{L+1}^x|V_{\mu\nu}=\nu id$. We can embed V into V_{long} via $v\mapsto v^{+}\otimes v\otimes v^{+}$, where $v^{+}=\binom{1}{1}$ is the eigenvector of σ_x with eigenvalue of 1. The isomorphic image of V is V_{+}. Under this embedding we see that $H\mapsto H_{\text{long}}|_{V++}$ and hence the spectrum of H is that of $H_{\text{long}}|_{V_{++}}$. The left inverse to the embedding map is the projection map $\pi: V_{\text{long}}\to V$ which simply projects out the first and last tensor factors.

B. Femionization of the extended chain

After substituting Eq. (5) into Eq. (13), the matrix M of Eq. (9) becomes

$$M=\begin{pmatrix} 0 & G & & & & \\ -G^T & 0 & F & & & \\ & -F^T & 0 & F & & \\ & & \ddots & \ddots & \ddots & \\ & & & -F^T & 0 & K \\ & & & & -K^T & 0 \end{pmatrix}, \tag{14}$$

where $F=1/4\begin{pmatrix} 0 & i \\ -i & 0\end{pmatrix}$ and $G=1/2\begin{pmatrix} 0 & 2i/\sqrt{8} \\ 0 & 0\end{pmatrix}$ and $K=1/2\begin{pmatrix} 0 & (i/\sqrt{8})2\cos(\chi) \\ 0 & (i/\sqrt{8})2\sin(\chi)\end{pmatrix}$. The corresponding diagonalization problem can be solved explicitly, see Ref. 14.

C. Projection to the original chain

Notice that the projection/restriction to V_{++} is clear in the spin-chain picture but not so obvious in the fermion language. The basic idea is to find the ground state which is in V_{++} and act on it by operators b. Let $\bar{B} \subset Cl^{+}_{2L+2}$ be the subalgebra generated by $b_1, \ldots, b_{L+1}$, then $\bar{B} = \bar{B}^{\text{even}} \oplus \bar{B}^{\text{odd}}$. There are vectors $v^{\pm}$ which satisfy $\sigma_0^x v^{\pm} = \pm v^{\pm}$ and either (1) $\sigma_{L+1}^x v^{\pm} = \pm v^{\pm}$ or (2) $\sigma_{L+1}^x v^{\pm} = \mp v^{\pm}$. Accordingly either

$$V_{++} = B^{\text{even}} v^{+} \text{ or } V_{++} = B^{\text{odd}} v^{-}. \tag{15}$$

We refer to Ref. 14 for the details. There is something interesting going on here that we would like to point out, although we do not fully understand the situation as of yet. The fact that V_{++} is generated by even or odd excitations seems to suggest that we are actually dealing with modules over $Cl^{\text{even}}_{2L+2} \subset Cl_{2L+2} \subset Cl_{2L+4}$, where Cl_{2L+2} is generated by the a_i and $b_i: i=1, \ldots, L+1$. Now it is well known that $Cl^{\text{even}}_{2L+2} \cong Cl_{2L+1}$.[12,13] The dimension of the V_{++} is 2^L so it is tempting to conjecture that it is actually one of the spin representations and σ_{L+1}^x corresponds to the operator that distinguishes the two spin representations. This operator is $t = i^n \prod_{j=1}^{L+1} e_j$ in the standard notation. A quick calculation shows that $t \propto \prod_{j=1}^{L+1} \sigma_j^z$, so things are in reality a little more complicated, but the dimensional analysis and the fact that there are two representations distinguished by an operator with eigenvalues of ± 1 remain true. It would be interesting to find a complete representation theoretic explanation for the projection mechanism.

III. SUPERSELECTION SECTORS AND THE THERMODYNAMIC LIMIT

A. Operators commuting with *H*

Just as we derived nontrivial spin chains from free fermions and vice versa found free fermion representations of nontrivial spin chains, we can take operators which obviously commute with the free fermion Hamiltonian and transform them back to the spin chain to obtain nontrivial conserved quantities and hence superselection rules. One such operator is the total fermion number operator; another operator of this kind is treated in Ref. 18.

To exhibit this strategy, we will now apply these observations to the XX-model with boundaries discussed in the last section. Moreover, as we explain in the next paragraph, this operator is key to understanding the spectrum of H and its dependence on the parameter χ. Recalling that the fermions entering the original chain are those labeled by $1, \ldots, L+1$ (see Sec. II C), the total fermion number for the spin chain H is given by the projection of the operator:

$$\mathcal{F}_{\text{tot}}^{\text{long}} = \sum_{n=1}^{L+1} N_n = \sum_{n=1}^{L+1} b_n a_n. \tag{16}$$

We note that *a priori* it is not clear that this operator can actually be "projected." *A posteriori* this follows either from the explicit form, $\mathcal{F}_{\text{tot}}^{\text{long}} = \sigma_0^x \otimes \mathcal{F}_{\text{tot}} \otimes \sigma_{L+1}^x$, whose calculation we describe below, or from the results about the spectrum being given by an even or odd number of fermion excitations (see Sec. II C).

Denoting the entries of the matrix $\mathbf{\Phi} = (\mathbf{\Phi}^{+} \mathbf{\Phi}^{-})$ appearing in Eq. (11) by $(\phi_n^{\nu})_j^{\mu}$, that is, $(\phi_n^{\nu})_j^{\mu}$ with n and $\nu = \pm 1$ fixed, $\mu = \pm 1$, and $j = 0, \ldots, L+1$ is the eigenvector to the eigenvalue $\nu \Lambda_n$, we can write

$$\mathcal{F}_{\text{tot}}^{\text{long}} = \frac{1}{4} \sum_{n=1}^{L+1} \sum_{j,k=0}^{L+1} \sum_{\mu,\nu=\pm 1} (\phi_n^{-})_j^{\mu} (\phi_n^{+})_k^{\nu} \tau_j^{\mu} \tau_k^{\nu}. \tag{17}$$

Now the explicit form of the eigenvectors ϕ, which are known,[14] allows us to compute this expression projected to the short chain V in terms of the σ matrices of the original spin-chain picture (see Ref. 19 for the details),

$$\mathcal{F}_{\rm tot} = \frac{1}{4L+4} \sum_{\substack{j,k=1 \\ j+k \text{ odd}, j<k}}^{L} (-1)^{(j+k+1)/2} (\sigma_{j+1}^z \cdots \sigma_{k-1}^z) \times \Bigg[\Bigg(\frac{\sin(\chi(j-k)/(L+1))}{\sin(\pi(j-k)/(2L+2))} + \frac{\sin(\chi(j+k)/(L+1))}{\sin(\pi(j+k)/(2L+2))} \Bigg) (\sigma_j^x \sigma_k^y - \sigma_j^y \sigma_k^x) + \Bigg(\frac{\cos(\chi(j+k)/(L+1))}{\sin(\pi(j+k)/(2L+2))} \Bigg) (\sigma_j^y \sigma_k^y - \sigma_j^x \sigma_k^x) \Bigg] + \frac{1}{\sqrt{8}(L+1)} \Bigg\{ \sum_{\substack{k=1 \\ k \text{ odd}}}^{L-1} (-1)^{(k+1)/2} \Bigg[\frac{\cos(\chi k/(L+1))}{\sin(\pi k/(2L+2))} \sigma_1^z \sigma_2^z \cdots \sigma_{k-1}^z \sigma_k^x \Bigg] + \sum_{\substack{k=2 \\ k \text{ even}}}^{L} \Bigg[(-1)^{(L+k+2)/2} \frac{\cos(\chi k/(L+1))}{\cos(\pi k/(2L+2))} \sigma_k^x + (-1)^{(k+L)/2} \frac{\sin(\chi k/(L+1))}{\cos(\pi k/(2L+2))} \sigma_k^y \Bigg] \times (\sigma_{k+1}^z \cdots \sigma_{L-1}^z \sigma_L^z) \Bigg\}. \quad (18)$$

In Eq. (18), we have taken the projection to the original chain H by using the equation $\mathcal{F}_{\rm tot}^{\rm long} = \sigma_0^x \otimes \mathcal{F}_{\rm tot} \otimes \sigma_{L+1}^x$. The fact that $\mathcal{F}_{\rm tot}^{\rm long}$ has this special form is the *a fortiori* reason that indeed the total fermion number is a quantity that is well defined on the original chain.

Depending on which case one is in (see Sec. II C), the eigenvalues are either odd or even. It is clear that it would be impossible to find this operator, which is a novel conserved quantity corresponding to a hidden symmetry for the spin chain, relying solely on the spin-chain picture.

B. The thermodynamic limit

Since we have superselection rules for the state space, we can look at the partition function Z_m of H restricted to the sector with m fermions. The relevant eigenvalues are given by[14,19]

$$\Lambda_n = \frac{1}{2} \sin\left(\frac{\chi}{L+1} + \frac{(2n-1)}{L+1} \frac{\pi}{2} \right), \quad n = 1, \ldots, L+1. \quad (19)$$

Since $\lim_{L\to\infty}(2L\Lambda_n/\pi) = \chi/\pi + 1/2 + n - 1$, in the thermodynamic limit,[20] we obtain

$$\begin{aligned} \mathcal{Z}_m = \lim_{L\to\infty} \{ \operatorname{tr} z^{(L/\pi)\sum_{n=1}^{L+1} 2\Lambda_n N_n} \} &= z^{m\chi/\pi + m/2} \sum_{n_1,n_2,\ldots,n_m} z^{n_1 - 1 + n_2 - 1 + \cdots + n_m - 1} \\ &= z^{m\chi/\pi + m/2} \left(\sum_l p_m(l) z^l + \sum_l p_{m-1}(l) z^l \right) \\ &= z^{m\chi/\pi + m/2} \left(\frac{z^{m(m+1)/2} + (1 - z^m) z^{(m-1)m/2}}{(1-z)(1-z^2)\cdots(1-z^m)} \right) \\ &= \frac{z^{m\chi/\pi + m^2/2}}{(1-z)(1-z^2)\cdots(1-z^m)}. \end{aligned} \quad (20)$$

Here $p_m(l)$ counts the number of ways the integer l can be expressed as a sum of m distinct nonzero integers. The second term involving p_{m-1} takes into account that one of $n_1 - 1$ in the sum before might be zero and the second equality follows from the formula

$$\sum_l p_m(l) z^l = \frac{z^{m(m+1)/2}}{(1-z)(1-z^2)\cdots(1-z^m)}. \quad (21)$$

The complete partition function $\mathcal{Z}$ will then be a sum over all the even or the odd m depending on χ. On the other hand $\mathcal{Z}$ has been calculated,[21] so that equating the two expressions, one obtains an interesting combinatorial identity.

What we gain from the calculation of $\mathcal{Z}_m$ is that we now know that in the sector with fixed fermion number, the dependence of $\mathcal{Z}_m$ on χ is given by a factor of $z^{m\chi/\pi}$. This simply induces a uniform shift off the usual spectrum. Notice that the factor is different though for different m, so that the eigenspaces of the thermodynamic limit of the operator $\mathcal{F}_{\text{tot}}$ play a special role, since the shift can precisely be factored out on these spaces. The reason for this factorization is the special distribution of the eigenvalues (19).

Another way to phrase this result is as follows. Consider the polynomial ring in infinitely many fermionic variables which are indexed by half integers $F=\mathbb{C}[\xi_i]:i\in\mathbb{N}_0+\frac{1}{2}$. This space comes equipped with a natural bigrading. The first grading is by the usual degree where $\deg(\xi_{i_1}\cdots\xi_{i_m})=m$. The second is by the weight wt where $\text{wt}(\xi_{i_1}\cdots\xi_{i_m})=\sum_{j=1}^m i_j$. Thus we can decompose $F=\oplus_d F_d$ according to the degree or according to the weight $F=\oplus_w F^w$. We will set F_d^w to be the bigraded piece of pure degree d and pure weight w and we will let π_d denote the projection of F onto its summand F_d. It is an elementary calculation to show that the dimension of $F_m^{l+m/2}$ is $p_m(l)+p_{m-1}(l)$ and likewise the dimension of $F_m^{l-m/2}$ is just $p_m(l)$.

Furthermore there are the two subspaces F_{even} and F_{odd} spanned by polynomials whose degree is either even or odd. Now the state space of H in the thermodynamic limit is abstractly isomorphic to one of the two subspaces F_{even} or F_{odd} of F (see Sec. II C). The isomorphism is given by sending the state $|n_1,\ldots,n_m\rangle$, which is the state with fermions n_i, to the monomial $\xi_{n_1-1/2}\cdots\xi_{n_m-1/2}$. (Recall that the n_i are positive integers and the indices of the ξ start at $0+\frac{1}{2}$.) In this language, the partition sum is the partition function $\text{tr}\, z^{\omega(\chi)}$ for the operator $\omega(\chi) := (\Sigma_m((m\chi/\pi)id+\text{wt})\circ\pi_m)|F_{\text{odd/even}}$. So we see that the eigenspaces are exactly the $F_m^{l+m/2}$ with the eigenvalues given by $m\chi/\pi+m/2+l$ and hence χ tunes the eigenvalue uniformly in each of the given sectors.

Now, in the original spin chain, this essential grading operator is given by the limit of Eq. (18). More details are contained in Ref. 19.

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