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Integrable quantum field theories with $OSP(m/2n)$ symmetries

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Abstract

We conjecture the factorized scattering description for $OSP(m/2n)/OSP(m-1/2n)$ supersphere sigma models and $OSP(m/2n)$ Gross–Neveu models. The non-unitarity of these field theories translates into a lack of 'physical unitarity' of the S-matrices, which are instead unitary with respect to the non-positive scalar product inherited from the orthosymplectic structure. Nevertheless, we find that formal thermodynamic Bethe ansatz calculations appear meaningful, reproduce the correct central charges, and agree with perturbative calculations. This paves the way to a more thorough study of these and other models with supergroup symmetries using the S-matrix approach.

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1. Introduction

The field theory approach to phase transitions in disordered systems has realized major progress over the last few years, thanks to an ever deeper understanding of two-dimensional field theories. Conformal invariance, combined with elegant reformulations using supersymmetry [1–3], and a greater control of non-unitarity issues [4–6], now severely constrains the possible fixed points [7,8]. In some simple cases, perturbed conformal field theory, combined with the use of current algebra symmetries, has even led to complete solutions [5,9]. Some of the models of interest in the context of disordered systems have also appeared independently in string theory [10,11], and more progress can only be expected from the cross fertilization between these two areas.

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Remarkably, the chief non-perturbative method, the integrable approach, has not been pushed very far to study these models. This is a priori surprising. For instance, several disordered problems involve variants of the $OSP(m/2n)$ Gross–Neveu model, which formally appears just as integrable as its well known $O(N)$ counterpart. The standard way of proceeding to study such a model would be to determine its S-matrix, and then use the thermodynamic Bethe ansatz and form-factors to calculate physical properties. This approach was pioneered in the elegant papers [12,13], and revived in [14], but so far the subject was only touched upon in our opinion; for instance, although the S-matrix of the $OSP(2/2)$ Gross–Neveu model has been conjectured [14], no calculation to justify this conjecture has been possible. Super sigma models have also been tackled, this time in the context of string theory [15], but there again results have only been very partial, and the S-matrix approach even less developed than for super Gross–Neveu models.

The main reasons for this unsatisfactory situation seem technical. While there has been tremendous progress in the understanding of the sine-Gordon model and the $O(3)$ sigma models—the archetypes of integrable field theories—models based on other Lie algebras are only partially understood (see [16,17] for some recent progress), and the situation becomes even more confusing when it comes to superalgebras. One of the main difficulties in understanding these theories is physical, and related with a general lack of unitarity—a feature that is natural from the disordered condensed matter point of view, but confusing at best from a field theory stand point. Another difficulty is simply the complexity of the Bethe ansatz for higher rank algebras, in particular, superalgebras. While these equations can be written sometimes (see the recent tour de force [18]), finding the pattern of solutions—the generalized string hypothesis—is a daunting task even for the trained expert [19].

Integrable field theories and lattice models go hand in hand, and the foregoing confusion seems to extend to spin chains based on superalgebras. Although the formalism is by now well in place to write the integrable Hamiltonians, their continuum limit is not well understood. In the case of ordinary algebras for instance, it is known that this continuum limit is a Wess–Zumino model on the group: whether this is true or not for superalgebras has been a matter of some debate [20]. Note that in some cases, the super spin chain is better understood than the field theory: this is the case for instance of the $sl(2/1)$ spin chain of [21,22] in the spin quantum Hall effect, whose relation to the traditional (super) Yang–Baxter formalism is also not understood at the present time.

Our purpose in this paper is to develop the integrable approach for the case of $OSP(m/2n)$ field theories. We will discuss two kinds of models, the supersphere sigma models, and the Gross–Neveu models, mostly for algebras $OSP(1/2n)$. In each case, we will conjecture a scattering theory, whose striking feature will be the lack of unitarity of the S matrices, as a result of the supergroup symmetry. We will argue that formal thermodynamical calculations do make sense nevertheless, and illustrate this point for both types of models.

2. Algebraic generalities

There are two basic integrable models with $O(N)$ symmetry, the Gross–Neveu model and the sphere sigma model $S^{N-1} = O(N)/O(N-1)$. Once their integrability is proven, the scattering theory is determined by implementing the action of the symmetry on the space of particles, and by requiring factorization. This is not always an obvious task, because of issues of bound states and charge fractionalization. For instance, the scattering theory for the $O(2P+1)$ Gross–Neveu model was completed only very recently [23]. However, the scattering of particles in the defining representation has been known for a long time [24], and this is where we would like to start here.

Scattering matrices with $O(N)$ symmetry can generally be written in terms of three independent tensors

$$\check{S}_{i_1 j_1}^{j_2 i_2} = \sigma_1 E + \sigma_2 P + \sigma_3 I, \tag{1}$$

where we have set

$$E_{i_1 j_1}^{j_2 i_2} = \delta_{i_1 j_1}\delta^{i_2 j_2}, \qquad P_{i_1 j_1}^{j_2 i_2} = \delta_{i_1}^{i_2}\delta_{j_1}^{j_2}, \qquad I_{i_1 j_1}^{j_2 i_2} = \delta_{i_1}^{j_2}\delta_{i_2}^{j_1} \tag{2}$$

corresponding to the graphical representation in Fig. 1.

We are interested here in models for which none of the amplitudes vanish. Specifically, for N a positive integer, there are generically two known models whose scattering matrix for the vector representation has the form (1), with none of the σ_i's vanishing. They are given by

$$\sigma_1 = -\frac{2i\pi}{(N-2)(i\pi-\theta)}\,\sigma_2, \qquad \sigma_3 = -\frac{2i\pi}{(N-2)\theta}\,\sigma_2 \tag{3}$$

with two possible choices for σ_2:

$$\sigma_2^{\pm}(\theta) = \frac{\Gamma\left(1-\frac{\theta}{2i\pi}\right)}{\Gamma\left(\frac{\theta}{2i\pi}\right)} \frac{\Gamma\left(\frac{1}{2}+\frac{\theta}{2i\pi}\right)}{\Gamma\left(\frac{1}{2}-\frac{\theta}{2i\pi}\right)} \frac{\Gamma\left(\pm\frac{1}{N-2}+\frac{\theta}{2i\pi}\right)}{\Gamma\left(1\pm\frac{1}{N-2}-\frac{\theta}{2i\pi}\right)} \frac{\Gamma\left(\frac{1}{2}\pm\frac{1}{N-2}-\frac{\theta}{2i\pi}\right)}{\Gamma\left(\frac{1}{2}\pm\frac{1}{N-2}+\frac{\theta}{2i\pi}\right)}. \tag{4}$$

The factor σ_2^+ does not have poles in the physical strip for $N \geqslant 0$, and the corresponding S-matrix for $N \geqslant 3$ is believed to describe the $O(N)/O(N-1)$ sphere (S^{N-1}) sigma model. The factor σ_2^- does not have poles in the physical strip for $N \leqslant 4$. For $N > 4$, it describes the scattering of vector particles in $O(N)$ Gross–Neveu model. Recall that for $N = 3, 4$ the vector particles are unstable and disappear from the spectrum, that contains only kinks. Some of these features are illustrated for convenience in Fig. 2.

Note that at vanishing rapidity, the scattering matrix reduces to $\check{S}(\theta = 0) = \mp I$. This is in agreement with the fundamental particles being bosons in the sigma model, and fermions in the Gross–Neveu model [25].

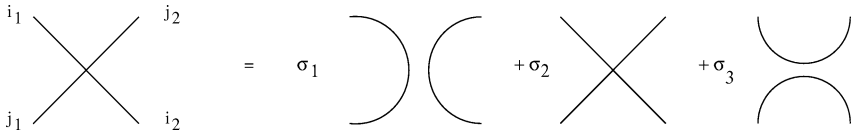

Fig. 1. Graphical representation of the invariant tensors appearing in the S-matrix.

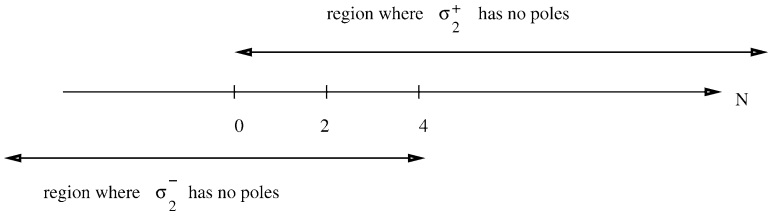

Fig. 2. Pole structure of σ_2 as a function of N.

Our next step is to try to define models for which $N < 1$, in particular $N = 0$, or N negative. A similar question has been tackled by Zamolodchikov [26] under the condition that particles be "impenetrable", that is $\sigma_1 = 0$. The (standard) procedure he used was to study the algebraic relations satisfied by the objects E, I for integer N, extend these relations to arbitrary N, and find objects (not necessarily $N \times N$ matrices) satisfying them. In technical terms, the algebraic relations turned out to be the defining ones for the Temperley–Lieb algebra [27], for which plenty of representations were known. The most interesting $N = 0$ case (corresponding to polymers) could then be studied using the 6-vertex model representation. It could also be studied using algebras $OSP(2n/2n)$, or algebras $GL(n/n)$.

In trying to address the same question for models where $\sigma_1 \neq 0$, it is natural to set up the problem in algebraic terms again. The objects E, P, I can be understood as providing a particular representation of the following Birman–Wenzl [28] algebra, defined by generators E_i, P_i, $i = 1, \ldots$ and relations

$$\begin{aligned} &P_i P_{i\pm1} P_i = P_{i\pm1} P_i P_{i\pm1}, \qquad P_i^2 = 1, \\ &[P_i, P_j] = 0, \quad |i-j| \geqslant 2, \end{aligned} \tag{5}$$

together with

$$\begin{aligned} &E_i E_{i\pm1} E_i = E_i, \qquad E_i^2 = N E_i, \\ &[E_i, E_j] = 0, \quad |i-j| \geqslant 2, \end{aligned} \tag{6}$$

and

$$\begin{aligned} &P_i E_i = E_i P_i = E_i, \\ &E_i P_{i\pm1} P_i = P_{i\pm1} P_i E_{i\pm1} = E_i E_{i\pm1}. \end{aligned} \tag{7}$$

These relations can be interpreted graphically as in Fig. 3; operators E define a sub-Temperley–Lieb algebra [27].

The natural extension of what was done say for polymers would be to look for vertex representations of the Birman–Wenzl algebra. However, this does not seem possible. The point is that the full Birman–Wenzl algebra has two parameters, and the representation furnished say by the spin one vertex model will have, for instance, that $P_i \neq P_i^{-1}$. This is a property natural from the knot theory framework where this algebra comes from, but disastrous for the construction of physical S matrices, where particles cannot "go under" another. Extending the definition of the S-matrix to arbitrary values of N thus seems problematic.

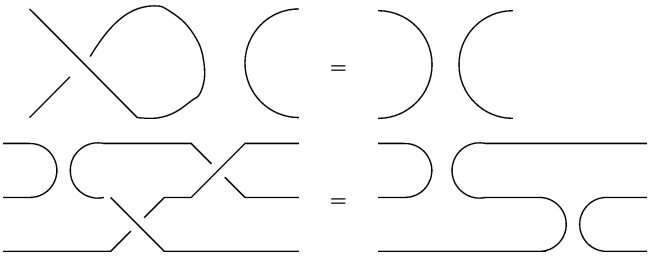

Fig. 3. Graphical representation for the defining relations of the BW algebra.

It is easy nevertheless to extend it to negative integer values of N. Indeed, the Birman–Wenzl algebras arise from representation theory of $O(N)$, and most of the properties of these algebras generalize to the superalgebras $OSP(m/2n)$ [29]. Instead of the vector representation of $O(N)$, take the vector representation of the orthosymplectic algebra, of dimensions $(m, 2n)$. For $m \neq 2n$, the tensor product with itself gives rise to three representations. Taking I as the identity, E as $(m-2n)$ times the projector on the identity representation, and P as the graded permutation operator (the extension to the case $m = 2n$ is easy), it can be checked indeed that the relations (5)–(7) are obeyed with $N = m - 2n$. More explicitly, in the usual case, the matrix elements of E are obtained by contracting the ingoing and outgoing indices using the unit matrix. In the *OSP* case, they are obtained similarly by contracting indices using the defining form of the *OSP* algebras

$$J = \begin{pmatrix} I_m & 0 & 0 \\ 0 & 0 & -I_n \\ 0 & I_n & 0 \end{pmatrix}. \tag{8}$$

In formulas, we set $\bar{\bar{i}} = i, i = 1, \ldots, m$, $\bar{i} = n + i, i = m+1, \ldots, m+n$, $\bar{\bar{i}} = i$. We set $x(i) = 1, i = m+1, \ldots, m+n$, $x(i) = 0$ otherwise, so $p(i) = x(i) + x(\bar{i})$. One has then

$$E_{i_1 j_1}^{j_2 i_2} = \delta_{i_1, \bar{j}_1} \delta^{i_2, \bar{j}_2} (-1)^{x(i_1)} (-1)^{x(i_2)} \tag{9}$$

while the graded permutation operator is of course given by

$$P_{i_1 j_1}^{j_2 i_2} = (-1)^{p(i_1)p(j_1)} \delta_{i_1}^{i_2} \delta_{j_1}^{j_2}. \tag{10}$$

This realization of Birman–Wenzl algebras was first mentioned in the very interesting paper [20]. It thus follows that the natural orthosymplectic generalization of the $\check{S}$-matrix of the $O(N)$ Gross–Neveu model (or sphere sigma model) does provide a solution of the Yang–Baxter equation, and realizes algebraically the continuation to values of N equal to zero or negative integers. Let us now discuss how meaningful this can be physically.

For this, let us recall some basic features about Yang–Baxter versus graded Yang–Baxter. In all cases, the Yang–Baxter formalism deals with two related objects that are usually called R, $\check{R}$ in a general context, S, $\check{S}$ in the context of scattering theory, and differ by some (graded) permutations.

In the ordinary case, we reserve the unchecked symbol to the matrix obeying

$$R_{12}(u-v) R_{13}(u) R_{23}(v) = R_{23}(v) R_{13}(u) R_{12}(u-v),$$

where u, v are spectral parameters. The equivalent of this relation for the superalgebra case is the graded Yang–Baxter equation, and it involves signs [30]:

$$R_{i_1 i_2}^{k_1 k_2}(u-v)R_{k_1 i_3}^{j_1 k_3}(u)R_{k_2 k_3}^{j_2 j_3}(v)(-1)^{p(i_1)p(i_2)+p(k_1)p(k_3)+p(k_2)p(k_3)}$$
$$= R_{i_2 i_3}^{k_2 k_3}(v)R_{i_1 k_3}^{k_1 j_3}(u)R_{k_1 k_2}^{j_1 j_2}(u-v)(-1)^{p(i_2)p(i_3)+p(i_1)p(k_3)+p(k_1)p(k_2)}, \tag{11}$$

where $p(k) = 1, 0$ is the parity of the k coordinate. These signs occur because, in the graded tensor product formalism, R_{13} acts on the first and third components, hence giving rise to potential minus signs when commuting through the elements of the second component. An ordinary (super) R-matrix does not solve the graded (ordinary) Yang–Baxter equation. However, if R does solve the graded Yang–Baxter equation, the object $\widetilde{R}_{ij}^{kl} \equiv (-1)^{p(i)p(j)}R_{ij}^{kl}$ solves then the ordinary Yang–Baxter equation, so it is easy to go from one point of view to the other.

In the ordinary case, one can also consider the object $\check{R} = PR$, P the permutation operator: this is what we gave in formula (1) for the case N a positive integer. It satisfies a different relation, $\check{R}_{12}(u-v)\check{R}_{23}(u)\check{R}_{12}(v) = \check{R}_{23}(v)\check{R}_{12}(u)\check{R}_{23}(u-v)$. Observe that this relation now involves only neighboring spaces in the tensor product, and thus is insensitive to grading. If R were to solve the graded Yang–Baxter equations instead, the same relation would be obeyed by the matrix $\check{R} = PR$, where now P is the graded permutation operator. Whether R satisfies the ordinary or the graded Yang–Baxter equation, it follows that matrices $\check{R}$ do satisfy the same equation. Conversely, a solution of $\check{R}_{12}\check{R}_{23}\check{R}_{12} = \check{R}_{23}\check{R}_{12}\check{R}_{23}$ can be interpreted as arising from a graded or a non-graded structure. The graded Yang–Baxter equation appears more as an aesthetically appealing object than a fundamental one. It is especially nice because it admits a classical limit, and fits in the general formalism of quantum supergroups [31].

In the context of scattering theories, which are our main interest here, it is convenient to define the S-matrix through the Fadeev–Zamolodchikov algebra [32]. Theories based on supergroups will have a spectrum of particles containing both bosons and fermions. Their creation and annihilation operators will be denoted $Z^{(\dagger)}$, and obey for instance $Z_i^\dagger(\theta_1)Z_j^\dagger(\theta_2) = (-1)^{p(i)p(j)}S_{ij}^{kl}(\theta_1-\theta_2)Z_l^\dagger(\theta_2)Z_l^\dagger(\theta_1)$. The consistency of these relations requires that S satisfies the graded Yang–Baxter equation, or, equivalently, that $\widetilde{S}$ satisfies the ordinary Yang–Baxter equation. Amplitudes of physical processes are then derived in the usual way. An important feature is that the monodromy matrix, which describes scattering of a particle through others, is built out of $\widetilde{S}$ like in the non-graded case (the same thing happens for integrable lattice models [31]).

Taking therefore our *OSP* $\check{S}$-matrix, and the S-matrix that follows from it, $S = \sigma_1 E + \sigma_2 I + \sigma_3 P$, it is natural to ask about the physical meaning of these amplitudes. This reveals some surprises. Crossing and unitarity are well implemented in the cases when the particles are bosons or fermions. Mixing the two kinds does not seem, a priori, to give rise to any difficulty. For instance the relation $S(\theta)S(-\theta) = \check{S}(\theta)\check{S}(-\theta) = I$ holds in the graded case with proper choice of normalization factors. It will turn out however that in the graded case, the S-matrix is, as a matrix, *not unitary*.[1] It is thus difficult to interpret our S-matrices

[1] This is a stronger violation of unitarity than in cases like the Lee–Yang singularity, where $SS^\dagger = 1$ still holds, but unphysical signs appear in S-matrix residues. For a thorough discussion of unitarity issues see [33].

in terms of a 'physical' scattering. The most useful way to think of the S-matrices will probably be as an object describing the monodromy of wave functions, like in imaginary Toda theories [16,34]. Crossing follows then from $\check{S}(i\pi-\theta)=\sigma_1(\theta)I+\sigma_2(\theta)P+\sigma_3(\theta)E$, with an obvious graphical interpretation, and charge conjugation being defined through the defining form of the OSP algebra.

Leaving aside the unitary difficulty, the usual formal procedure thus selects once again the factors $\sigma_2^{\pm}$ as minimal prefactors, with the continued values $N=m-2n$. The question is then to establish the relation what field theory, if any.

Obvious candidates are the $OSP(m/2n)$ Gross–Neveu model with action (in all this paper, normal ordering is left implicit)

$$S=\int\frac{d^2x}{2\pi}\Bigg[\sum_{i=1}^{m}\psi_L^i\partial\psi_L^i+\psi_R^i\bar{\partial}\psi_R^i+\sum_{j=1}^{n}\beta_L^j\partial\gamma_L^j+\beta_R^j\bar{\partial}\gamma_R^j + g\big(\psi_L^i\psi_R^i+\beta_L^j\gamma_R^j-\gamma_L^j\beta_R^j\big)^2\Bigg], \tag{12}$$

where the ψ are Majorana fermions of conformal weight $1/2$, and the $\beta\gamma$ are bosonic ghosts of weight $1/2$ as well. Perturbative calculations of the beta function [3,35] suggest that this model behaves like the continuation of the $O(N)$ Gross–Neveu model to the value $N=m-2n$. Similarly, the natural generalization of the sphere sigma model is a super sphere sigma model, which can be described as the coset $OSP(m/2n)/OSP(m-1/2n)$. There again, perturbative beta functions do match. It is therefore natural to expect that the S-matrices built on $OSP(m/2n)$ will describe, depending on the prefactor $\sigma_2^{\pm}$, these two models in the appropriate physical regimes. This will be discussed in the next section.

3. The $OSP(1/2)$ sigma model S-matrix

3.1. The S-matrix

To make things more concrete, let us discuss the case $N=-1$, and its realization using $OSP(1/2)$. Instead of the Gross–Neveu model, it will turn out to be easier to study the equivalent of the sigma model, because of its relation with the $a_2^{(2)}$ Toda theory and spin chain.

The solution of the graded Yang–Baxter equation relevant here is the well known $OSP(1/2)$ one, given by

$$R_{OSP(1/2)}=\frac{1}{1-3\frac{\theta}{2i\pi}}\left[P+\frac{3\theta}{2i\pi}I+\frac{\theta}{i\pi-\theta}E\right], \tag{13}$$

where we have chosen the normalization factor for later purposes, I is the identity. Denote the basis vectors in the fundamental representation of $OSP(1/2)$ as b, f_1, f_2. The operator E is given by the matrix

$$E=\begin{pmatrix}1&-1&1\\1&-1&1\\-1&1&-1\end{pmatrix} \tag{14}$$

in the subspace spanned by (b,b), (f_1,f_2), (f_2,f_1) in that order, $E=0$ otherwise. In that same subspace, the graded permutation operator reads

$$P=\begin{pmatrix}1&0&0\\0&0&-1\\0&-1&0\end{pmatrix}. \tag{15}$$

The operators E,P satisfy the defining relations of the Birman–Wenzl algebra with $N=-1$.

The non-graded $\check{R}$ matrix meanwhile reads

$$\check{R}_{OSP(1/2)}=\frac{1}{1-3\frac{\theta}{2i\pi}}\left[I+\frac{3\theta}{2i\pi}P+\frac{\theta}{i\pi-\theta}E\right]. \tag{16}$$

Let us now discuss the issue of unitarity. While $R(\theta)R(-\theta)=\check{R}(\theta)\check{R}(-\theta)=1$, R, $\check{R}$, and $\widetilde{R}$ as matrices, are unitary only with respect to an indefinite metric induced by the supergroup structure. Explicitly, one has for instance

$$\check{R}_{bb}^{bb}\check{R}_{bb}^{*bb}-\check{R}_{bb}^{f_1f_2}\check{R}_{bb}^{*f_1f_2}-\check{R}_{bb}^{f_2f_1}\check{R}_{bb}^{*f_2f_1}=1 \tag{17}$$

and in fact $\check{R}$ conserves a scalar product that allows for negative norm square states $\langle ff|ff\rangle=-1$, all others equal to $+1$. It is well-known indeed [36] that the structure of $OSP(1/2)$ is not compatible with a positive scalar product. The mere presence of supergroup symmetry leads necessarily to the existence of negative norm-square states, and therefore to unitarity problems.

The resulting scattering matrix is therefore non-unitary, in the usual sense. This is a consequence of the orthosymplectic supergroup symmetry, and originates physically in the non-unitarity of the field theory described by the S-matrix. This does not prevent one from using the S-matrix at least to describe the monodromy of the wave functions, as we will do in the section devoted to TBA. Similarly, this S-matrix could also be used to describe aspects of the finite size spectrum [33,34].

An intriguing remark is that, although the matrix $\widetilde{R}$ is not unitary, its eigenvalues happen to be complex numbers of modulus one (the same hold for R and $\check{R}$),[2] and there are reasons to believe that this is true for the eigenvalues of the monodromy matrices involving an arbitrary number of particles. This means that the non-unitarity situation is not as stringent as say in the $a_2^{(1)}$ case [16], and that, for instance, the spectrum of the theory in finite size will be real.

Let us now consider the 'scattering' theory that is the continuation of the sphere sigma model to $N=-1$: we take the $OSP(1/2)$ realization, and as a prefactor σ_2^+. It then turns out that the S-matrix is identical to the one of the $a_2^{(2)}$ Toda theory for a particular value of the coupling constant! This will allow us to explicitly perform the TBA, and identify the scattering theory indeed. While we were carrying out these calculations, we found out two papers where the idea has been carried out to some extent already: one by Martins [37], and one by Sakai and Tsuboi [38]. Our approach has little overlap with these papers, and stems from our earlier work on the $a_2^{(1)}$ theory instead.

[2] We thank G. Takacs for suggesting this may be the case.

To proceed, we now discuss the $a_2^{(2)}$ Toda theory in more details.

3.2. A detour through $a_2^{(2)}$

This theory has action

$$S = \frac{1}{8\pi}\int dx\,dy\left[(\partial_x\Phi)^2 + (\partial_y\Phi)^2 + \Lambda\left(2e^{-\frac{i}{\sqrt{2}}\beta\Phi} + e^{i\sqrt{2}\beta\Phi}\right)\right]. \tag{18}$$

The conformal weight of the first field is $\Delta_1 = \beta^2/4$, while the one of the second is $\Delta_2 = \beta^2$. The dimension of Λ is such that $[\Lambda]^3 L^{-2h_2-4h_1} = L^{-6}$, so $[\Lambda] = L^{-2}L^{\frac{2h_2+4h_1}{3}}$, i.e., $[\Lambda] = L^{\beta^2-2}$, and the "effective" dimension (i.e., twice the conformal weight) of the perturbation is $d = \beta^2$.

The domain we shall be interested in primarily corresponds to $\beta^2 \geqslant 1$. We will parameterize

$$\beta^2 = 2\frac{t-1}{t} \tag{19}$$

so $h_1 = \frac{\beta^2}{4} = \frac{t-1}{2t}$, $[\Lambda] = L^{-2/t}$. The case $t = 2$ corresponds to $h_1 = 1/4$, and the limit $t \to \infty$ to $h_1 = 1/2$.

The massless or massive nature of the theory depends on the sign of Λ and on the value of β^2 [39]. For $\beta^2 \leqslant 1$, the theory is massive for $\Lambda < 0$, but for the region we are interested in, $\Lambda > 0$ is required, and we will restrict to this in the following.

In the $t \in [2,\infty]$ domain, the scattering matrix has been first conjectured by Smirnov [40]. The spectrum does not contain any bound states, and is simply made of solitons with topological charges $\pm 1, 0$ (where the topological charge is defined as $q = \frac{1}{2\sqrt{2}\pi\beta}\int \partial_x\phi$). The relation between the mass of the solitons and the coupling constant reads [39]

$$\Lambda^3 = -\frac{1}{16\pi^3}\frac{\Gamma^2(\beta^2/4)\Gamma(\beta^2)}{\Gamma^2(1-\beta^2/4)\Gamma(1-\beta^2)}\left[\frac{\pi M}{\sqrt{3}\,\Gamma(1/3)}\frac{\Gamma(2/3(2-\beta^2))}{\Gamma\left(\beta^2/3(2-\beta^2)\right)}\right]^{3(2-\beta^2)}. \tag{20}$$

Near $\beta^2 = 2$, which will turn out to be the point with $OSP(1/2)$ symmetry, setting $\beta^2 = 2-\epsilon$, one has $\Lambda^3 \propto \epsilon M^{3\epsilon}$.

The $\check{S}$-matrix is proportional to the $\check{R}$-matrix of the Izergin–Korepin model [41]. Although this may seem laborious, we will write it explicitly here. Introducing the parameter

$$\xi = \frac{2}{3}\frac{\pi\beta^2}{2-\beta^2} \tag{21}$$

and the variables $\lambda = e^{-2\pi\theta/5\xi}$, $p = e^{i\pi/2}e^{i\pi/3\xi}$, we write

$$\check{S} = \Sigma_0\frac{1}{\lambda^5 p^5 - \lambda^{-1}p^{-5} + p^{-1} - p}\check{R}$$

with [41,42]

$$
\begin{aligned}
&\check{R}^{11}_{11} = \check{R}^{-1,-1}_{-1,-1} = \lambda^5 p^5 - \lambda^{-1} p^{-5} + p^{-1} - p,\\
&\check{R}^{0,1}_{1,0} = \check{R}^{1,0}_{0,1} = \check{R}^{-1,0}_{0,-1} = \check{R}^{0,-1}_{-1,0} = \lambda p^3 - \lambda^{-1} p^{-3} + p^{-3} - p^3,\\
&\check{R}^{-1,1}_{1,-1} = \check{R}^{1,-1}_{-1,1} = \lambda p - \lambda^{-1} p^{-1} + p^{-1} - p,\\
&\check{R}^{00}_{00} = \lambda p^3 - \lambda^{-1} p^{-3} + p^{-3} - p^3 + p^{-1} - p + p^5 - p^{-5},\\
&\check{R}^{01}_{01} = \check{R}^{-1,0}_{-1,0} = \lambda\big(p^5 - p\big) + p^{-1} - p^{-5},\\
&\check{R}^{-1,1}_{0,0} = \check{R}^{0,0}_{-1,1} = \lambda\big(p^4 - 1\big) + 1 - p^4,\\
&\check{R}^{1,0}_{1,0} = \check{R}^{0,-1}_{0,-1} = \lambda^{-1}\big(p^{-1} - p^{-5}\big) + p^5 - p,\\
&\check{R}^{0,0}_{1,-1} = \check{R}^{1,-1}_{0,0} = \lambda^{-1}\big(1 - p^{-4}\big) + p^{-4} - 1,\\
&\check{R}^{-1,1}_{1,-1} = \lambda\big(p^5 - p - p^3 + p^{-1}\big) + p^3 - p^{-5},\\
&\check{R}^{1,-1}_{1,-1} = \lambda^{-1}\big(p^{-1} - p^{-5} - p + p^{-3}\big) + p^5 - p^{-3}.
\end{aligned}
\tag{22}
$$

The normalization factor admits the representation

$$
\Sigma_0 = -\exp\left[i \int_{-\infty}^{\infty} \frac{d\omega}{\omega} e^{-3i\omega\theta/\pi} \frac{\sinh(3\omega)\cosh(3\omega(2\xi-\pi)/4)}{\sinh(3\omega\xi/2\pi)\cosh(3\omega/2)} \right]. \tag{23}
$$

It is equal to the amplitude for the scattering processes $11 \to 11$.

In the case $\xi \to \infty$, one checks that

$$
\frac{1}{\lambda^5 p^5 - \lambda^{-1} p^{-5} + p^{-1} - p} \check{R} \longrightarrow \check{R}_{osp(1/2)} \tag{24}
$$

(with $b \leftrightarrow 0$, $f_{1,2} \leftrightarrow \pm 1$) up to an irrelevant gauge transformation. Moreover, it turns out that

$$
\Sigma_0 \frac{\frac{3\theta}{2i\pi}}{1 - \frac{3\theta}{2i\pi}} \longrightarrow \sigma_2^+ \tag{25}
$$

or $\Sigma_0 = \sigma_3^+ - \sigma_2^+$ for $N = -1$, confirming the identification of the $a_2^{(2)}$ $\check{S}$-matrix in the limit $t \to \infty$ with the $OSP(1/2)$ "sphere sigma model" $\check{S}$-matrix.

This coincidence has a simple algebraic origin. Indeed recall [43,44], that the $a_2^{(2)}$ Toda theory has symmetry $U_q\big(a_2^{(2)}\big)$, $q = e^{i\pi/\beta^2}$. The Dynkin diagram for the algebra $a_2^{(2)}$ turns out to be almost identical to the one for the algebra $osp(1|2)^{(1)}$ [29], as represented in Fig. 4, although in the latter case, one of the roots is fermionic, and therefore the basic relations involve an anticommutator instead of a commutator.

It can be hoped that for some particular value of q, the q-deformation of one algebra gives rise to the other, and this is what we shall now demonstrate—namely, that there is a mapping between $U_q\big(a_2^{(2)}\big)$ and $U(osp(1|2)^{(1)})$, for $q = i$. This should not come as a surprise, and has algebraic roots going back as far as [45]. For recent related works, see [46,47].

Fig. 4. Dynkin diagrams for $a_2^{(2)}$ and $osp(1/2)^{(1)}$.

Traditionally, the Cartan matrix of $a_2^{(2)}$ is written as $\left(\begin{smallmatrix} 8 & -4 \\ -4 & 2 \end{smallmatrix}\right)$, and the commutation relations are

$$\begin{aligned} &[H_i, H_j] = 0, \qquad [H_i, E_j] = a_{ij} E_j, \\ &[H_i, F_j] = -a_{ij} F_j, \\ &[E_i, F_j] = \delta_{ij} \frac{q^{H_i} - q^{-H_i}}{q_i - q_i^{-1}}, \quad q_i = q^{a_{ii}/2}. \end{aligned} \tag{26}$$

This means, in particular, that the generators E_0, F_0, H_0 satisfy a $U_{q^4}(a_1)$ algebra

$$\begin{aligned} &[H_0, E_0] = 8E_0, \qquad [H_0, F_0] = -8F_0, \\ &[E_0, F_0] = \frac{q^{H_0} - q^{-H_0}}{q^4 - q^{-4}} \end{aligned} \tag{27}$$

while the generators E_1, F_1, H_1 satisfy a $U_q(a_1)$ algebra

$$\begin{aligned} &[H_1, E_1] = 2E_1, \qquad [H_1, F_1] = -2F_1, \\ &[E_1, F_1] = \frac{q^{H_1} - q^{-H_1}}{q - q^{-1}}. \end{aligned} \tag{28}$$

The Cartan matrix of $osp(1|2)^{(1)}$ on the other hand reads usually $\left(\begin{smallmatrix} 4 & -2 \\ -2 & 1 \end{smallmatrix}\right)$. Commutation relations are similar to (26), but involve anticommutators instead of commutators for the fermionic generators. The generators e_0, f_0, h_0 satisfy thus a a_1-algebra

$$\begin{aligned} &[h_0, e_0] = 4e_0, \qquad [h_0, f_0] = -4f_0, \\ &[e_0, f_0] = h_0, \end{aligned} \tag{29}$$

while for the generators $\psi_1^\dagger$, ψ_1, h_1 one has

$$\begin{aligned} &\left[h_1, \psi_1^\dagger\right] = \psi_1^\dagger, \qquad [h_1, \psi_1] = -\psi_1, \\ &\left\{\psi_1^\dagger, \psi_1\right\} = h_1. \end{aligned} \tag{30}$$

Taking $q = i$ for $U_q\left(a_2^{(2)}\right)$ makes the subalgebra generated by E_0, F_0, H_0 and $U(a_1)$ algebra. The value $q = i$ for the other deformed a_1 was already observed in [46] to allow a simple relation with a fermionic algebra, a fact also used in mapping $U_i(a_1)$ onto a supersymmetric $\mathcal{N} = 1$ algebra. Here, observe that by setting $\psi_1^\dagger = q^{-(H_1+1)/2} E_1$ and $\psi_1 = q^{(H_1-1)/2} F_1/(q + q^{-1})$, one finds, *for representations where H_1 is even* (the only

ones of interest in our case), that, when $q \to i$,

$$\left\{\psi_1^\dagger, \psi_1\right\} = \frac{H_1}{2}$$

in agreement with the anticommutation relation for $U(osp(1|2))$ if $h_1 = H_1/2$. The rest of the relations then are in complete agreement, up to some straightforward changes of normalization.

We conclude that, restricting to representations with H_1 even, the two algebras are isomorphic. Since this constraint is satisfied in the case at hand, the $osp(1/2)^{(1)}$ symmetry of the $a_2^{(2)}$ Toda theory is thus explained.

3.3. Thermodynamic Bethe ansatz

Throughout this paper, we will use the thermodynamic Bethe ansatz to calculate physical properties of our theory. It is a priori unclear whether the method—which involves maximizing a free energy—makes much sense in a theory whose Hamiltonian is not Hermitian, but the results we obtain seem perfectly meaningful, like in other similar examples. Two additional remarks about the TBA are relevant. First, the scattering matrix appearing in the auxiliary monodromy problem (diagonalizing the matrix describing the effect of passing a particle through the others) is not S but $\widetilde{S}$. This means that, although the S-matrices of the $osp^{(1)}$ and $a_2^{(2)}$ differ because of the grading, the objects used in the TBA (like the $\widetilde{S}$ matrices) are identical, and known results about $a_2^{(2)}$ Toda theories can be used. Second, one may worry that mixing bosons and fermions could give rise to problems in applying the TBA. This is not quite so however. Most TBAs known so far—and the ones we will introduce here will be no exceptions—allow at most one particle in a state of a given rapidity. As discussed in Zamolodchikov [48], this corresponds, in the diagonal case, to having $S_{ii}^{ii}(0) = -(-1)^F$, where F is the fermion number of particle i. In our case, we have $S_{ii}^{ii}(0) = \mp P_{ii}^{ii}$. For the supersphere sigma model, the particles with bosonic internal labels $i = 1, \ldots, m$, will be bosons, so $P_{ii}^{ii} = (-1)^F$. For the super Gross–Neveu model, the particles with bosonic internal labels are now fermions, so $P_{ii}^{ii} = -(-1)^F$. In both cases, the required result holds.

The TBA analysis can be performed using the well known strategies. The only difficulty is the diagonalization of the monodromy matrix, which involves solving an auxiliary problem based on the $a_2^{(2)}$ vertex model. String solutions for this model were not known before, but they can easily be obtained using our recent results on the $a_2^{(1)}$ case. Setting $\gamma = \frac{\pi}{t-1}$, the $a_2^{(2)}$ Bethe equations have the form

$$\prod_\alpha \frac{\sinh\frac{1}{2}(y_i - u_\alpha - i\gamma)}{\sinh\frac{1}{2}(y_i - u_\alpha + i\gamma)} = \prod_j \frac{\sinh\frac{1}{2}(y_i - y_j - 2i\gamma)}{\sinh\frac{1}{2}(y_i - y_j + 2i\gamma)} \frac{\sinh\frac{1}{2}(y_i - y_j + i\gamma)}{\sinh\frac{1}{2}(y_i - y_j - i\gamma)}, \quad (31)$$

where the y_i are Bethe roots, and the u_α are spectral parameter heterogeneities (corresponding to the rapidities of particles already present in the system). The solutions of these equations in the thermodynamic limit are as follows. The y's can be $1, 2, \ldots, t-1$, strings,

or antistrings. In addition, it is possible to have a t string centered on an antistring, or to have a complex of the form $y = y_r \pm \frac{i\gamma}{2} + i\pi$.

After the usual manipulations, one ends up with equations for the pseudoenergies, that can be represented using a TBA diagram. The 'left part' of the diagram corresponds to the following equations

$$\frac{\epsilon_j(\theta)}{T} = \phi_3(\theta-\theta') \star \ln\bigl(1+e^{\epsilon_j(\theta')/T}\bigr) - \sum_{l=0}^{t-3} (\delta_{j,l+1}+\delta_{j,l-1})\phi_3(\theta-\theta') \star \ln\bigl(1+e^{-\epsilon_l(\theta')/T}\bigr), \tag{32}$$

where we denote $\phi_P(\theta) = \frac{P}{2\cosh(P\theta/2)}$, $f \star g(\theta) = \int_{-\infty}^{\infty} \frac{d\theta'}{2\pi} f(\theta-\theta')g(\theta')$. We use in the following the Fourier transform

$$\hat{f}(\omega) = \int \frac{d\theta}{2\pi} e^{iP\omega\theta/\pi} f(\theta) \tag{33}$$

so $\widehat{(f \star g)} = 2\pi \hat{f}\hat{g}$, and $\hat{\phi}_P = \frac{1}{2\cosh\omega}$. We introduce the other kernel ψ defined by $\hat{\psi} = \frac{\cosh\omega/2}{\cosh\omega}$.

In addition, there is a set of equations providing a closure on the right part.

$$\frac{\epsilon_{t-3}(\theta)}{T} = \phi_3(\theta-\theta') \star \ln\bigl(1+e^{\epsilon_{t-3}(\theta')/T}\bigr) - \phi_3(\theta-\theta') \star \ln\bigl(1+e^{-\epsilon_{t-4}(\theta')/T}\bigr) - \sum_{i=1}^{3} \phi_3(\theta-\theta') \star \ln\bigl(1+e^{-\epsilon_{a_i}(\theta')/T}\bigr) - \psi(\theta-\theta') \star \ln\bigl(1+e^{-\epsilon_b(\theta')/T}\bigr). \tag{34}$$

Together with

$$\frac{\epsilon_{a_i}(\theta)}{T} = -\phi_3(\theta-\theta') \star \ln\bigl(1+e^{-\epsilon_{t-3}(\theta')/T}\bigr) + \phi_3(\theta-\theta') \star \ln\bigl(1+e^{\epsilon_{a_i}(\theta')/T}\bigr) + \sum_{j\neq i} \phi_3(\theta-\theta') \star \ln\bigl(1+e^{-\epsilon_{a_j}(\theta')/T}\bigr) + \psi(\theta-\theta') \star \ln\bigl(1+e^{-\epsilon_b(\theta')/T}\bigr) \tag{35}$$

and

$$\frac{\epsilon_b(\theta)}{T} = -\psi(\theta-\theta') \star \ln\bigl(1+e^{-\epsilon_{t-3}(\theta')/T}\bigr) + 2\phi_3(\theta-\theta') \star \ln\bigl(1+e^{\epsilon_b(\theta')/T}\bigr) + \sum_{i=1,2} \psi(\theta-\theta') \star \ln\bigl(1+e^{-\epsilon_{a_i}(\theta')/T}\bigr) + \psi(\theta-\theta') \star \ln\bigl(1+e^{\epsilon_{a_3}(\theta')/T}\bigr). \tag{36}$$

Finally, the asymptotic conditions $\epsilon_0(\theta\to\infty) \to m\cosh\theta$ must be imposed. This system can be conveniently encoded in the diagram of Fig. 5. The free energy per unit length reads as usual

$$F = -T \int_{-\infty}^{\infty} \frac{d\theta}{2\pi} m\cosh\theta \ln\bigl(1+e^{-\epsilon_0/T}\bigr). \tag{37}$$

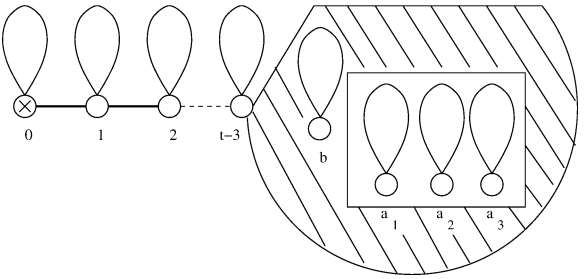

Fig. 5. Incidence diagram for the TBA of the (anisotropic) $a_2^{(2)}$ theory. Nodes are associated with the pseudoenergies ϵ, and the cross indicates the presence of a massive asymptotic behavior for ϵ_0.

We will consider the more general case of twisted boundary conditions, by adding a phase factor in the trace $Z = \mathrm{Tr}[e^{-\beta H} e^{i\alpha q/(t-1)}]$, q the topological charge. The kinks have therefore a fugacity $(e^{\pm i\alpha/t-1}, 1)$. We concentrate on the central charge, which is expressed in terms of the quantities $x = e^{-\epsilon/T}$ in the limits of large and small temperature. At large temperature (UV), the x_j go to constants, which solve the following system (here we set $\lambda = e^{i\alpha}$, which appears in the equations due to a renormalization of the spin [49]):

$$
\begin{aligned}
&x_0 = (1+x_1)^{1/2}\left(1+\frac{1}{x_0}\right)^{-1/2},\\
&\cdots\\
&x_n = (1+x_{n-1})^{1/2}(1+x_{n+1})^{1/2}\left(1+\frac{1}{x_n}\right)^{-1/2},\\
&\cdots\\
&x_{t-3} = (1+x_{t-4})^{1/2}(1+x_a)^{1/2}(1+\lambda x_a)^{1/2}\left(1+\lambda^{-1}x_a\right)^{1/2}\\
&\qquad\times\left(1+\frac{1}{x_{t-3}}\right)^{-1/2}(1+x_b),\\
&x_a = (1+x_{t-3})^{1/2}\left(1+\frac{1}{x_a}\right)^{-1/2}(1+\lambda x_a)^{-1/2}\left(1+\lambda^{-1}x_a\right)^{-1/2}(1+x_b)^{-1},\\
&x_b = (1+x_{t-3})\left(1+\frac{1}{x_a}\right)^{-1}(1+\lambda x_a)^{-1}\left(1+\lambda^{-1}x_a\right)^{-1}(1+x_b)^{-1},
\end{aligned}
\tag{38}
$$

and recall that there are three (like the dimension of the fundamental representation) nodes with a common value of x_a. The solution of this system is

$$
\begin{aligned}
&x_j = \frac{\sin\frac{(j+1)\alpha}{2t}\sin\frac{(j+4)\alpha}{2t}}{\sin\frac{\alpha}{2t}\sin\frac{\alpha}{t}}, \quad j = 0,\ldots,t-3,\\
&x_a = \frac{\sin\frac{(t-1)\alpha}{2t}}{\sin\frac{(t+1)\alpha}{2t}}, \qquad x_b = \frac{\sin^2\frac{(t-1)\alpha}{2t}}{\sin\frac{\alpha}{t}\sin\alpha}.
\end{aligned}
\tag{39}
$$

What we will in general call the UV contribution to the central charge is $c_1 = \frac{6}{\pi^2}\sum L\left(\frac{x_j}{1+x_j}\right)$. At small temperature (IR), the x_j go similarly to constants solving the same system but with one less node on the left, because $x_0 \to 0$. We first consider the case

$\alpha = 0$, i.e., periodic boundary conditions for the bosons, antiperiodic boundary conditions for the fermions. In that case, the UV sum of dilogarithms gives a contribution $(t-1)$, while one gets a similar contribution from the IR sum after $t \to t-1$, $c_2 = t-2$. The central charge is thus $c = c_1 - c_2 = 1$, as expected.

(Here we include two specialized remarks:

A point of some interest is $\gamma = \pi/2$, corresponding to $h_1 = 1/4$. In that case, the $a_2^{(2)}$ Bethe equations do coincide (after a shift $y \to y + \frac{i\pi}{2}$) with the $a_1^{(1)}$ Bethe equations that appear in solving the sine-Gordon model with $\frac{\beta_{\mathrm{SG}}^2}{8\pi} = 1/4$. This point is in the attractive regime, with one soliton and one antisoliton of mass m, and one breather of the same mass. It is easy to check that in that case, the $a_2^{(2)}$ TBA is in fact identical with the well known SG TBA indeed. The equivalence between the two theories is not so obvious when one looks at the actions.

Also, as $a_2^{(2)}$ is related to $a_2^{(1)}$, so does the $a_2^{(2)}$ theory bear some resemblance to the $a_2^{(1)}$ Toda theory with the following action

$$S = \frac{1}{8\pi} \int dx\, dy \left[\sum_{i=1}^{2} (\partial_x \phi_i)^2 + (\partial_y \phi_i)^2 - \Lambda' \left(e^{i\frac{\beta}{\sqrt{2}}(\phi_1 + \sqrt{3}\phi_2)} + e^{i\frac{\beta}{\sqrt{2}}(\phi_1 - \sqrt{3}\phi_2)} + e^{-i\sqrt{2}\beta\phi_1} \right) \right]. \tag{40}$$

Here the perturbation has a single dimension $h = \beta^2$, and the dimension of the coupling is $[\Lambda] = L^{2\beta^2 - 2}$. Parameterizing β in (40) by $\beta^2 = \frac{t-1}{t}$, it turns out that the free energy of the $a_2^{(2)}$ theory is exactly half the free energy of the $a_2^{(1)}$ theory, once the fundamental masses have been matched. This fact does not appear obvious in the least when one compares perturbative expansions!)

Twistings and truncations of the $a_2^{(2)}$ model are of the highest interest and have been widely discussed in the literature [40,42,44]. Twisting (that is, putting a charge at infinity) in such a way that $e^{i\beta\phi}$ becomes a screening operator of weight $\Delta = 1$, gives the central charge $c = 1 - \frac{3(t-2)^2}{(t-1)t}$. RSOS restriction is then possible for t even, giving rise to the minimal model $M_{t-1,t/2}$. The perturbation in the minimal model has then weight $\Delta_{21} = 1 - \frac{3}{2t}$ (its coupling is real, and the sign does not matter because it has only even non-vanishing correlators). Meanwhile, the lowest weight $\Delta_{12} = \frac{4-t}{8(t-1)}$ becomes negative for $t \geqslant 4$, after which the effective central charge reads $c_{\mathrm{eff}} = c - 24h_{12} = 1 - \frac{12}{t(t-1)}$. One can also twist in such a way that $e^{-i\beta\phi/2}$ becomes a screening operator, giving the central charge $c = 1 - \frac{3(t+2)^2}{(t-1)t}$. RSOS restriction is then possible for t odd, giving rise to $M_{t,(t-1)/2}$ perturbed by the operator of weight $\Delta_{15} = 1 - \frac{3}{t}$.[3]

[3] Notice that the combination $2 - x = 3/t$, respectively, $6/t$ for t even (respectively, odd). In fact, the perturbative series for the free energy always has the same structure, and does not exhibit parity effects as t is changed. But the physical interpretation does, and rightly so, since for ϕ_{21}, only even correlation functions do not vanish, while for ϕ_{15}, all correlation functions are a priori non-vanishing.

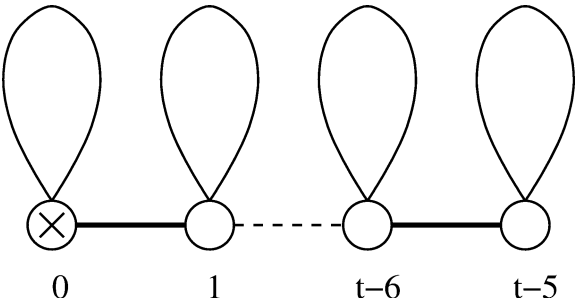

Fig. 6. Incidence diagram of the truncated $a_2^{(2)}$ TBA. This TBA also describes perturbations of the $OSP(1/2)_{(4-t)}/SU(2)_{(t-4)/2}$ models for t even, see later.

The twisting can also be studied with the TBA using now $\alpha \neq 0$. The UV sum of dilogarithms gives then a contribution $(t-1) - 3\frac{t-1}{t}\frac{\alpha^2}{\pi^2}$, while one gets a similar contribution from the IR sum after $t \to t-1$. The central charge is thus $c = 1 - \frac{3}{t(t-1)}\frac{\alpha^2}{\pi^2}$.

In general, twisting terms affecting nodes 'far to the right' of the TBA diagram do not affect the central charge in the isotropic limit. Indeed, if α were to remain finite here as $t \to \infty$, the central charge of the twisted theory would be still $c = 1$. We shall however be interested in giving antiperiodic boundary conditions to the kinks of charge $q = \pm 1$, which translates into a phase that blows up like $\alpha \approx t\pi$ as $t \to \infty$. As a result, the central charge of interest is $c \to -2$, in agreement with the sigma model interpretation to be discussed next.

Finally, we notice that choosing $\alpha = 2\pi$ leads to $x_{t-4} = 0$, and a truncation of the diagram to the one represented in Fig. 6. This is the same as folding the TBA for the $a_2^{(1)}$ RSOS model with central charge $c = 2 - \frac{24}{(t-1)t}$. The first model in the series has $c_{\text{eff}} = 2/5$, the next one $c_{\text{eff}} = 3/5$ (the latter TBA has some fascinating properties, due to the fact that $2/5 + 3/5 = 1$). This was first observed in [50]. We will comment about the relation of these models to $OSP(1/2)$ in the conclusion. For TBAs related with $a_2^{(2)}$ in other regimes, see [51,52].

3.4. The OSP(1/2) limit, and the relation with the sigma model

As explained previously, the $OSP(1/2)$ scattering theory can be studied by taking the $t \to \infty$ limit of the $a_2^{(2)}$ model. The identification could in fact be seen directly by identifying Bethe equations. This seems a bit strange at first, because the $a_2^{(2)}$ equations do not have a structure that is reminiscent of the $osp(1/2)$ Dynkin diagram. One has to remember however that the $osp(1/2)$ Bethe ansatz equations are peculiar, and their structure is not related with the Cartan matrix in the usual way. They read in fact [18]

$$\prod \frac{\lambda_i - \mu_\alpha - i}{\lambda_i - \mu_\alpha + i} = \prod \frac{\lambda_i - \lambda_j - 2i}{\lambda_i - \lambda' + 2i} \prod \frac{\lambda_i - \lambda_j + i}{\lambda_i - \lambda_j - i} \tag{41}$$

and match the $a_2^{(2)}$ equations in the $t \to \infty$ limit, with $y = \gamma\lambda$, $u = \gamma\mu$, $\gamma \to 0$.

The Toda theory (18) can then be rewritten in terms of a Dirac fermion as

$$S = \int \frac{d^2x}{2\pi}\Big[\psi_R^\dagger \partial \psi_R + \psi_L^\dagger \bar{\partial}\psi_L + \Lambda\big(\psi_R^\dagger \psi_L + \psi_R \partial\psi_R \psi_L^\dagger \bar{\partial}\psi_L^\dagger\big)\Big] \tag{42}$$

the perturbation is the sum of a term of dimension $h_1 = 1/2$, and a term of dimension $h_2 = 2$ (the relative normalization between the two fermionic terms is irrelevant, since it can be adjusted by $\psi_R \to \lambda\psi_R$, $\psi_R^\dagger \to \lambda^{-1}\psi_R^\dagger$, or similarly for left fermions). It is likely that this model could be directly diagonalized using the coordinate Bethe ansatz, like the ordinary massive Thirring model, but we have not carried out such a calculation. Conserved quantities can be found in terms of the fermions; the first ones are $\psi_R^\dagger$, $\psi_R^\dagger \partial(\partial\psi_R^\dagger\psi_R)\ldots$.

The twisted theory meanwhile has $c = -2$, $c_{\text{eff}} = 1$, and the perturbations both acquire dimensions $(1,1)$. This can be identified with a symplectic fermion theory with action [53]

$$S = \int \frac{d^2x}{2\pi}\big[\partial_\mu\eta_1\partial_\mu\eta_2 + \Lambda'\partial_\mu\eta_1\partial_\mu\eta_2 + \Lambda''\eta_1\eta_2\partial_\mu\eta_1\partial_\mu\eta_2\big]. \tag{43}$$

Here η_1 and η_2 are two fermionic fields with propagator, in the free theory,

$$\big\langle \eta_1(z,\bar{z})\eta_2(0)\big\rangle = -\ln z\bar{z}.$$

Notice how non-unitarity is manifest in (42) as well as (43).

From the point of view of the twisted theory, the perturbation involves two fields of weights $(1,1)$ which should be identified with ϕ_{21} and ϕ_{15}, respectively, using fusion relations. That both fields appear is not unexpected, since the $c = -2$ point is a limit, and should have the characteristics of both t-even and t-odd.

The identification of $\partial_\mu\eta_1\partial_\mu\eta_2$ with ϕ_{21} can actually be completed accurately, by comparing the four point functions as calculated in the fermion theory and the minimal model using the Dotsenko–Fateev general results [54]. An interesting sign subtlety appears in that case. Indeed, $\partial_\mu\eta_1\partial_\mu\eta_2 = \bar{\partial}\eta_1\partial\eta_2 + \partial\eta_1\bar{\partial}\eta_2$, and if we call this operator O,

$$\big\langle O(1)O(2)\big\rangle = -\frac{1}{z_{12}^2\bar{z}_{12}^2}$$

because of anticommutation relations. Hence, $\partial_\mu\eta_1\partial_\mu\eta_2$ should actually be identified with $i\phi_{21}$. In fact, when one compares the amplitude of the perturbation in the $a_2^{(2)}$ Toda theory and the twisted version [39], one finds that, with the usual normalizations, Λ positive gives rise to the amplitude of ϕ_{21} being purely imaginary, that is the coefficient Λ' in (43) real. The sign of Λ' is irrelevant, as only terms even in Λ' will appear in the perturbative expansions of physical quantities.

It would be very interesting to complete the identification of ϕ_{15} with $\eta_1\eta_2\partial_\mu\eta_1\partial_\mu\eta_2$, but we have not finished this calculation. Note, however, that there is little doubt this identification is correct, as there is no other object with the right dimension and statistics in the symplectic fermion theory. Defining $O = \eta_1\eta_2\partial_\mu\eta_1\partial_\mu\eta_2$, one finds

$$\big\langle O(1)O(2)\big\rangle = \frac{1 + (\ln|z_{12}|^2)^2}{|z_{12}|^2}.$$

The massive perturbation with ϕ_{15} is obtained with a coefficient that is real and positive near $\beta^2 = 2$ [39]. Therefore, we expect Λ'' in (43) to be positive. Note that the apparition of

logarithms in the two point function of the perturbing operator makes the field theory (43) a bit problematic. Issues of renormalizability arise, in particular, and it is probably better to think of (43) as a sector of (42) rather than the defining theory. This is reflected in the structure of the TBA: although $c=-2$ can formally be obtained as the UV value of the central charge in the untwisted model, this value appears only after proper analytic continuation of the dilogarithms. Indeed, the fugacity given to end nodes of the TBA diagram is $e^{i\alpha}$, and as $t\to\infty$, $\alpha\approx t\pi$, it winds an infinite number of times around the origin: in practice, following the free energy would presumably require following analytic continuations on an infinity of different branches, a difficult task at best.

The fermions can always be rescaled to bring the action into the form

$$S=\int\frac{d^2x}{2\pi}[\partial_\mu\eta_1\partial_\mu\eta_2+\Lambda\eta_1\eta_2\partial_\mu\eta_1\partial_\mu\eta_2], \tag{44}$$

where again the coupling Λ is positive. We will now see how this related to the supersigma model.

In general, the coset space $OSP(m/2n)/OSP(m-1/2n)$ has dimensions $(m-1,2n)$ and can be interpreted as the supersphere $S^{m-1,2n}$ [6]. The case of interest here is $m=n=1$, and corresponds to the $S^{0,2}$ supersphere, parameterized by the coordinates

$$x_1=1-\frac{1}{2}\eta_1\eta_2,\qquad \xi_1=\eta_1,\qquad \xi_2=\eta_2 \tag{45}$$

such that $x_1^2+\xi_1\xi_2=1$. The action of the sigma model will generally be of the form

$$\frac{1}{g}\left(\sum_{i=1}^{m}(\partial_\mu x_i)^2+\sum_{j=1}^{n}\partial_\mu\xi_{2j-1}\partial_\mu\xi_{2j\,j}\right)$$

(our convention is that the Boltzmann weight is e^{-S}). The beta function will be to first order $\beta\propto(m-2n-2)g^2$, so for the region $m-2n<2$ in which we are interested, the model will be free in the UV and massive in the IR for a *negative* coupling constant, $g=-|g|$. In the $S^{0,2}$ case, this action therefore reads

$$S=-\frac{1}{|g|}\int d^2x\left[\partial_\mu\eta_1\partial_\mu\eta_2-\frac{1}{2}\eta_1\eta_2\partial_\mu\eta_1\partial_\mu\eta_2\right]. \tag{46}$$

Note that a rescaling combined with a relabeling can always bring this action into the form

$$S=\int d^2x\left[\partial_\mu\eta_1\partial_\mu\eta_2+\frac{|g|}{4\pi}\eta_1\eta_2\partial_\mu\eta_1\partial_\mu\eta_2\right] \tag{47}$$

matching the $t\to\infty$ limit of the $a_2^{(2)}$ theory, with $\Lambda\propto|g|$.

4. Supersphere sigma models and integrable superspin chains

The relation we uncovered between $a_2^{(2)}$ and $OSP(1/2)$ extends immediately to the case of $a_{2n}^{(2)}$ and $OSP(1/2n)$: one can establish, for general values of n, the relation between the quantum affine algebras, the Bethe ansatz equations, the scattering matrices etc. We thus

propose that the S-matrix with $OSP(1/2n)$ symmetry, represented in (1), (3), (4) with $N = 1-2n$, and the prefactor σ_2^+, provides an analytic continuation of the $O(N)/O(N-1)$ "sphere" sigma model to this value of N.

Of course, the analytic continuation of the sigma model should be interpreted as the coset $OSP(1/2n)/OSP(0/2n)$. The effective central charge of the UV limit is $c_{\text{eff}} = n$, while its true central charge will be $c = -2n$. For the ordinary sigma models, the UV central charge is $N-1$, so the UV value in the analytic continuation just matches.

The $a_{2n}^{(2)}$ Toda theory has an interaction term of the form

$$e^{\frac{i}{\sqrt{2}}\beta(\phi_1-\phi_2)} + e^{\frac{i}{\sqrt{2}}\beta(\phi_2-\phi_3)} + \cdots + e^{\frac{i}{\sqrt{2}}\beta(\phi_{n-1}-\phi_n)} + e^{i\sqrt{2}\beta\phi_n}.$$

The dimension of vertex operators $\exp\bigl(i\sum\delta_j\phi_j\bigr)$ is $h = \sum\frac{1}{2}\delta_j^2 - \delta_0\sum\delta_j$, where δ_0 measures the twist, and the central charge is $c = n(1-12\delta_0^2)$. The operators in the interaction term all have $h=1$ when $\delta_0 = 1/2$, and $\beta = \sqrt{2}$. In that case, $c = -2n$, while all the operators $e^{\pm\frac{i}{\sqrt{2}}(\phi_j-\phi_k)}$ and $e^{2i\phi_j}$ have dimensions $(1,1)$.

The manifold relevant for $OSP(1,2n)$ on the other hand is $S^{0,2n}$, i.e., a purely "fermionic sphere". For instance, $S^{0,4}$ can be parameterized by

$$\begin{aligned} &x_1 = 1 - \frac{1}{2}(\eta_1\eta_2 + \eta_3\eta_4) - \frac{1}{4}\eta_1\eta_2\eta_3\eta_4, \\ &\xi_1 = \eta_1, \qquad \xi_2 = \eta_2, \qquad \xi_3 = \eta_3, \qquad \xi_4 = \eta_4. \end{aligned} \tag{48}$$

The action of the sigma model is not particularly illuminating; it involves four and six fermions couplings, and reduces to $2n$ symplectic fermions in the UV limit. Like in the $n=1$ case, it can be matched onto the appropriate limit of the $a_{2n}^{(2)}$ theory.

On the other hand, it is also possible to extend the analysis of the $a_2^{(2)}$ TBA to arbitrary value of n, so we also know the TBA for this scattering theory, which is simply given by a Z_2 folding of the $a_{2n}^{(1)}$ TBA. The TBA is represented in Fig. 7.

Notice that there are n massive particles: while for N integer positive the $O(N)/O(N-1)$ S-matrix has no bound states, with simply N fundamental particles (in the vector representation), poles do enter the physical strip for $N<2$. For the value $N = 1-2n$ we are interested in, the masses of the particles are $m_i \propto \sin\frac{i\pi}{2n+1}$, $i = 1,\ldots,n$. The UV central charge is easily checked to be $c_{\text{eff}} = n$. We do not know how to obtain the central charge of the untwisted theory, as this would require a knowledge of the 'closure' of the TBA diagram for twisted theories, an unsolved problem when $n>1$.

Our results have an immediate application to the study of quantum spin chains. Indeed, the Bethe equations which appear in the solution of the $OSP(1/2n)$ sigma models are similar to the ones appearing in the solution of the integrable $OSP(1/2n)$ chains studied, in particular, by Martins and Nienhuis [20]. More detailed calculations show that these chains are critical, and that they coincide at large distance with the weakly coupled supersphere sigma models, that is, a system of $2n$ free symplectic fermions. This is in disagreement with the conjecture in [20,37] that this continuum limit should be a WZW model on the supergroup: although the central charge agrees with both proposals, detailed calculations of the thermodynamics or finite size spectra show that the WZW proposal is not correct, and confirm the sigma model proposal instead. A similar conclusion holds for $OSP(m/2n)$

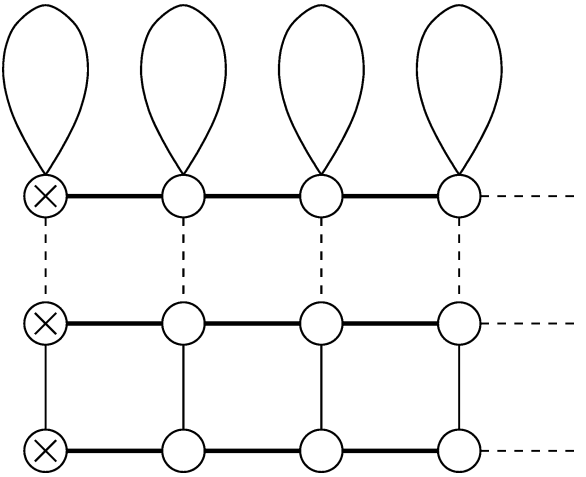

Fig. 7. Incidence diagram for the TBA of $OSP(1/2n)$ sigma models (the diagram has n rows).

when $m-2n<2$. That the spin chain flows to the weakly coupled sigma model is certainly related with the change of sign of the beta function when $m-2n$ crosses the value 2, but we lack a detailed understanding, similar to the ones proposed in [55,56], of the mechanisms involved.

5. The super Gross–Neveu models

5.1. Generalities

If we consider a scattering matrix defined again by (1), (3), but now with the prefactor σ_2^- instead, it is natural to expect that it describes $OSP(m/2n)$ Gross–Neveu models, the analytic continuation of the $O(N)$ GN models to $O(m-2n)$. Having a control on the diagonalization of $OSP(1/2n)$ scattering matrices will allow us to study this scattering theory easily, and confirm the identification for these algebras. Notice that since the $O(N)$ scattering matrix has no poles in the region $N<2$, the roles of the GN and sigma models are completely exchanged in the domain of values of N we are considering.

The $OSP(m/2n)$ Gross–Neveu models read

$$S=\int\frac{d^2x}{2\pi}\Bigg[\sum_{i=1}^{m}\psi_L^i\bar\partial\psi_L^i+\psi_R^i\bar\partial\psi_R^i+\sum_{j=1}^{n}\beta_L^j\bar\partial\gamma_L^j+\beta_R^j\bar\partial\gamma_R^j + g\big(\psi_L^i\psi_R^i+\beta_L^j\gamma_R^j-\gamma_L^j\beta_R^j\big)^2\Bigg]. \tag{49}$$

This theory has central charge $c=m/2-n$, effective central charge $c_{\text{eff}}=m/2+2n$. The beta function for this model is of the form $\beta_g\propto(m-2n-2)g^2$, the same as the one for the $O(m-2n)$ GN model. For $m-2n>2$, it is thus positive, so a positive

coupling g is marginally relevant—this is the usual massive GN model—while a negative one is marginally irrelevant. If instead we consider the case $m - 2n < 2$, these results are switched: it is a negative coupling that is marginally relevant, and makes the theory massive in the IR.[4] The case $m = 1$ should be described by the foregoing scattering theory.

Note that the GN model is equivalent to the appropriate WZW model with a current–current perturbation. Indeed, the system of m Majorana fermions and n symplectic bosons constitutes in fact a certain representation of the $OSP(m/2n)$ current algebra. The level depends on the choice of normalization; it would be called $k = 1/2$ in [57], $k = -1/2$ in [58], $k = 1$ elsewhere. We adopt the latter convention here, and thus the level k WZW model based on $OSP(m/2n)$ has central charge

$$c = \frac{(m-2n)(m-2n-1)k}{2(m-2n-2+k)}. \tag{50}$$

Particular cases are $OSP(0/2n)$, which coincides with the $SP(2n)$ WZW model at level $-k/2$, and $OSP(m/0)$, which coincides with the $O(m)$ WZW model at level k. Supersymmetric space theorems give rise to free fields representations at level $k = 1$, where $c = \frac{m-2n}{2}$, at level $k = m - 2n - 2$, where

$$c = \frac{(m-2n)(m-2n-1)}{4} = \frac{\operatorname{sdim} OSP(m/2n)}{2}.$$

Notice that the representation at level -2 for $OSP(2/2)$ described recently in [59] is a particular case of the supersymmetric space theorem discussed in Goddard et al. [57] (for $k = -1$ in their notations).

The $OSP(m/2n)$ Gross–Neveu models present additional non-unitarity problems not encountered in the sigma models discussed above. To tackle these problems, we first discuss the simplest case of all.

5.2. The $OSP(0/2)$ case

We consider the case of the GN model for $N = -2$, corresponding formally to $OSP(0/2)$, i.e., a $\beta\gamma$ system. The S-matrix should act on a doublet of particles, and reads, from the general formulas

$$\check{S} = \tan\left(\frac{\pi}{4} + \frac{i\theta}{2}\right)\frac{\Gamma\left(\frac{1}{2}+\frac{\theta}{2i\pi}\right)}{\Gamma\left(\frac{1}{2}-\frac{\theta}{2i\pi}\right)}\frac{\Gamma\left(-\frac{\theta}{2i\pi}\right)}{\Gamma\left(\frac{\theta}{2i\pi}\right)}\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & \frac{1}{\theta - i\pi} & -\frac{\theta}{\theta - i\pi} & 0 \\ 0 & -\frac{\theta}{\theta - i\pi} & \frac{1}{\theta - i\pi} & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{51}$$

It turns out that $\check{S} = i\tanh\left(\frac{\theta}{2} - \frac{i\pi}{4}\right)\check{S}_{\mathrm{SG}}(\beta_{\mathrm{SG}}^2 = 8\pi)$ where S_{SG} is the soliton S-matrix of the sine-Gordon model. At coupling $\beta_{SG}^2 = 8\pi$, it coincides with the S-matrix of the $SU(2)$ invariant Thirring model, or the level 1 WZW model with a current–current perturbation.

[4] In [14], the four fermion coupling is defined through combinations $\bar{\psi}_-\psi_+ + \psi_-\bar{\psi}_+ = 2i\left(\psi_L^1\psi_R^1 + \psi_L^2\psi_R^2\right)$, so what is called g there is the opposite of our convention.

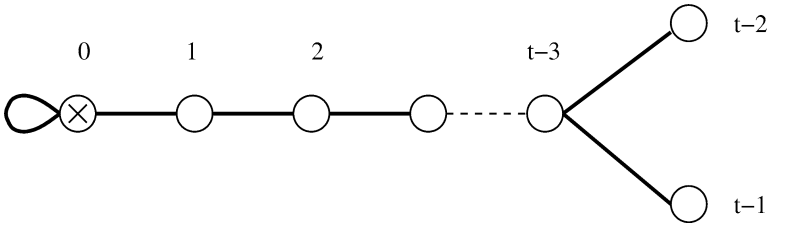

Fig. 8. Incidence diagram for the TBA describing the current–current perturbation of the (anisotropic) $\beta\gamma$ system (or $OSP(0/2)$) model.

The scattering matrix is thus the same as the one for the $k = 1$ $SU(2)$ WZW model up to a CDD factor. This CDD factor does not introduce any additional physical pole, but affects the TBA in an essential way.

Note that the $N = -2$ Gross–Neveu model can also be considered as a current–current perturbation of the $SU(2)$ WZW model at level $k = -1/2$, or the $SU(1,1)$ model at level $k = 1/2$.

To study the TBA, it is useful as in the sigma model case to consider the anisotropic deformation, with the sine-Gordon part now corresponding to $\frac{\beta_{SG}^2}{8\pi} = \frac{t-1}{t}$. The diagram is represented in Fig. 8. The UV solutions have to obey the usual SG equations plus the fact that

$$x_0 = (1 + x_1)^{1/2}(1 + x_0)^{1/2}. \tag{52}$$

The solution is obtained by setting $x_j = (j + \alpha)^2 - 1$, $j = 0, \ldots, t - 3$, $x_{t-2} = x_{t-1} = t - 3 + \alpha$, and letting $\alpha \to \infty$. The contribution to the central charge in the UV is then $c_1 = t$, the number of nodes. The solution in the IR is obtained by discarding the first node, and then coincides with the usual IR solution of the SG equations, with $\alpha = 1$. The contribution to the central charge is equal to $c_2 = t - 2$, the number of nodes *minus one*. The final central charge is thus $c = 2$, as expected for the effective central charge of the $\beta\gamma$ system.

The same calculation with a fugacity $e^{i\pm\frac{\alpha}{t-1}}$ gives $c = 2 - \frac{6\alpha^2}{(t-1)\pi^2}$. This is because in the UV, all the x's are still infinite, giving rise this time to $c_1 = t - \frac{6\alpha^2}{\pi^2}$, while in the IR, the x's are the same as the ones for the ordinary sine-Gordon model, with $c_2 = t - 2 - 6\frac{t-2}{t-1}\frac{\alpha^2}{\pi^2}$. This result requires explanations; in particular, setting $\alpha = (t-1)\pi$ and letting $t \to \infty$ as in the sigma model case gives $c = -\infty$!

5.3. The role of zero modes

We want to consider in more details the $\beta\gamma$ system with action

$$S = \int \frac{d^2x}{2\pi}(\beta_L \partial\gamma_L + \beta_R \bar{\partial}\gamma_R). \tag{53}$$

The propagators are

$$\gamma_R(z)\beta_R(w) = -\beta_R(z)\gamma_R(w) = \frac{1}{z-w}.$$

We can 'bosonize' the ghosts by introducing a scalar field $\Phi = \phi_R + \phi_L$, such that $\phi_R(z)\phi_R(w) = -\ln(z-w)$. We also introduce fermionic ghosts $\eta_R(z)\xi_R(w) = \xi_R(z)\eta_R(w) = \frac{1}{z-w}$, and thus

$$\gamma_R = e^{\phi_R}\eta_R, \qquad \gamma_L = e^{-\phi_L}\eta_L,$$
$$\beta_R = e^{-\phi_R}\partial\xi_R, \qquad \beta_L = e^{\phi_L}\bar{\partial}\xi_L. \tag{54}$$

The corresponding action is then

$$S = \frac{1}{8\pi}\int d^2x\,(\partial_\mu\Phi)^2 = \frac{1}{2\pi}\int d^2x\,\partial\Phi\bar{\partial}\Phi. \tag{55}$$

The $\beta\gamma$ Hamiltonian is

$$H = \frac{1}{4\pi}\int dx\,(\beta_L\partial_x\gamma_L + \beta_R\partial_x\gamma_R) \tag{56}$$

with commutators $[\beta_L(x),\gamma_L(y)] = \frac{i}{4\pi}\delta(x-y)$, $[\beta_R(x),\gamma_R(y)] = -\frac{i}{4\pi}\delta(x-y)$. The $U(1)$ current is given by $J_R = \gamma_R\beta_R = \partial\phi_R$, $J_L = \gamma_L\beta_L = -\partial\phi_L$. The topological charge is

$$Q = \frac{1}{2\pi}\int dx\,(J_R - J_L) = \frac{1}{2\pi}\int dx\,\partial_x\Phi. \tag{57}$$

The topological charge of γ_L and γ_R is 1, while the charge of β_L and β_R is -1.

A key feature of this system is the existence of zero modes. With periodic boundary conditions, it is indeed easy to see that $[H, \int dx\,\beta_{L,R}] = [H, \int dx\,\gamma_{L,R}] = 0$. It follows from this that, if we add to the Hamiltonian a term of the form $-hQ$, the system will fill up with an infinity of zero mode particles of β or γ type depending on the sign of h, sending the ground state energy to $-\infty$. The theory is thus unstable without a mass term. In the current algebra language, the infinite-dimensional space associated with the zero mode decomposes into lowest weight representations of $SU(1,1)_{1/2}$ of 'angular momentum' $j = -1/4$ and $j = -3/4$. The conformal weight of these states is $\Delta = -1/8$, giving the effective central charge $c_{\text{eff}} = 2$ for a $c = -1$ theory indeed.

The mass term (which is actually a current–current perturbation) in the $OSP(0/2)$ GN model does stabilize the theory. To see how, let us add to the action a term

$$\delta S = -\frac{h}{2\pi}\int d^2x\,(\gamma_R\beta_R - \gamma_L\beta_L) + \frac{g}{8\pi}\int d^2x\,(\gamma_R\beta_L - \gamma_L\beta_R)^2. \tag{58}$$

The classical minima of $S + \delta S$ occur for $\gamma_R = \gamma_L = c$ and $\beta_R = -\beta_L = b$, and, turning to the Hamiltonian formalism, the minimum energy becomes then

$$\frac{1}{L}E_{gs} = -\frac{1}{2\pi}\frac{h^2}{g}. \tag{59}$$

We now recall the RG equation for the coupling constant g in (58):

$$\dot{g} = -2g^2. \tag{60}$$

From this, the coupling constant at scale $1/m$ goes like $g = \frac{1}{2\ln cst/m}$. The constant term is a UV cut-off, provided here by the field h. It follows that

$$\frac{1}{L}E_{gs} = -\frac{1}{\pi}h^2 \ln(h/m) \tag{61}$$

at leading order. If $m \to 0$ ($g \to 0$), we recover the result $E_{gs} \to -\infty$ anticipated before.

We will comment more on the behavior of the $OSP(0/2)$ and other GN models later. For the moment, our goal is to explain the behavior of the central charge in the anisotropic case obtained in the previous section. So, we now consider the case where an anisotropy is imposed on the system by adding a coupling of the form $J_L J_R$. More explicitly, consider

$$\delta A = -\frac{h}{2\pi}\int d^2x\,(\gamma_R\beta_R - \gamma_L\beta_L) - \frac{g}{8\pi}\int d^2x\, 2\gamma_R\beta_R\gamma_L\beta_L. \tag{62}$$

It is easy to calculate the ground state energy at leading order as $g \to 0$, which turns out to be finite now, even though the theory is still massless: $E_{gs}/L = -\frac{h^2}{\pi g}$. Anisotropy has stabilized the UV theory.

We can now calculate this ground state energy using the S-matrix approach. To do so, we perturb the action by a term of the form $\beta_R^2\gamma_L^2 + \beta_L^2\gamma_R^2$. In the bosonized version, this reads

$$e^{-2\Phi}\bar\partial\eta_L\eta_L\partial^2\xi_R\partial\xi_R + e^{2\Phi}\partial\eta_R\eta_R\bar\partial^2\xi_L\bar\partial\xi_L.$$

The anisotropic term changes the kinetic term to $\frac{1}{8\pi}(1-g/2)(\partial_\mu\Phi)^2$. We can renormalize the field so the kinetic term looks as before, and then the exponentials in the perturbation become $e^{\pm 2\hat\beta\Phi}$ with $\hat\beta^2 = \frac{1}{1+g/2}$. Non-local conserved currents are then obtained using

$$\exp\left(\frac{-2\phi}{\hat\beta}\right)\partial^2\xi_R\partial\xi_R, \qquad \exp\left(\frac{2\phi}{\hat\beta}\right)\partial^2\eta_R\eta_R$$

of dimension $\Delta_c = 3 - 2/\hat\beta^2$. They lead to a quantum deformation of the $a_1^{(1)}$ algebra with quantum parameter $q = e^{-i\pi\Delta_c}$. Setting $q = e^{i\pi\frac{t-2}{t-1}}$, this corresponds to a thermodynamic Bethe ansatz diagram with t nodes (including the source one), i.e., the TBA studied in the previous section and represented in Fig. 8. The point is, we now have the correspondence between the coupling g in the anisotropic action and the parameter t in the anisotropic TBA, with, at small g or large t, $g \approx 1/t$. To use this TBA, we finally need to establish the correspondence between the magnetic field and the kinks fugacity: setting $e^{\pm h/T} = e^{i\pm\alpha/t-1}$, where T is the temperature, gives $h = \frac{Ti\alpha}{t-1}$. Using this, the TBA result $c = 2 - \frac{6}{\pi}\frac{\alpha^2}{t-1}$ does match the ground state energy $E_{gs} = -\frac{h^2}{\pi g}$ at leading order as $g \to 0$. We thus have explained the TBA results of the previous section in the light of the ground state instability of the $\beta\gamma$ system.

For the $\beta\gamma$ system itself, we of course obtain $c = -\infty$ for any non-trivial fugacity of the kinks. This can also be understood as follows: the partition function of the $\beta\gamma$ conformal field theory with periodic boundary conditions is infinite because of the presence of a bosonic zero mode [60]. On the other hand, in the periodic sector, the TBA gives a finite result, with the central charge $c = 1/2 + 2$. Therefore, the TBA approach must describe a renormalized partition function, divided by the infinite contribution of the bosonic zero

mode. In the antiperiodic sector where there is no such zero mode, the result of this division is to give zero, or, formally, a central charge equal to $-\infty$. Nevertheless, the dependence of the ground state energy on α can be predicted and checked using the TBA, which provides another non-trivial check of the S-matrix.

5.4. The $OSP(1/2n)$ case

We now get back to the $OSP(1/2n)$ case. The TBA turns out to have a simple description in terms of $a_{2n}^{(2)}$ again. Consider therefore, not the $SU(2n+1)$ GN model, but a related scattering theory with only two multiplets of particles, corresponding, respectively, to the defining representation and its conjugate. Considering more generally the case of $SU(P)$ models, the relation between the $SU(P)$ GN scattering theory and this new theory is similar to the relation between the $O(P)$ GN model and the $O(P)/O(P-1)$ sigma model [61]. We will thus call this scattering theory 'sigma model like', but we are not aware of any physical interpretation for it. The TBA equations can be written following the usual procedure. They read

$$\begin{aligned}
\frac{\epsilon_{aj}}{T} &= \sum_{b=1}^{P-1} I_{ab}^{(P)} \phi_P \star \ln\bigl(1+e^{\epsilon_{bj}/T}\bigr) - \sum_{l=1}^{\infty} I_{jl}^{(\infty)} \phi_P \star \ln\bigl(1+e^{-\epsilon_{al}/T}\bigr), \quad j \geqslant 2,\\
\frac{\epsilon_{a1}}{T} &= \sum_{b=1}^{P-1} I_{ab}^{(P)} \phi_P \star \ln\bigl(1+e^{\epsilon_{bj}/T}\bigr) - \phi_P \star \ln\bigl(1+e^{-\epsilon_{a2}/T}\bigr)\\
&\quad - (\delta_{a1}+\delta_{a,P-1})\phi_P \star \ln\bigl(1+e^{-\epsilon_{a0}/T}\bigr)
\end{aligned} \tag{63}$$

for the pseudoparticles, and

$$\begin{aligned}
\frac{\epsilon_{10}}{T} &= \frac{m\cosh\theta}{T} + \sum_{a=1}^{P-1} \phi_{1a} \star \ln\bigl(1+e^{\epsilon_{a1}/T}\bigr),\\
\frac{\epsilon_{P-1,0}}{T} &= \frac{m\cosh\theta}{T} + \sum_{a=1}^{P-1} \phi_{P-1,a} \star \ln\bigl(1+e^{\epsilon_{a1}/T}\bigr).
\end{aligned} \tag{64}$$

In these equations again, $\phi_P(\theta) = \frac{P}{2\cosh\frac{P\theta}{2}}$, $\phi_{P-1,a} = \phi_{1,P-a}$, and $\hat{\phi}_{1a}(\omega) = \frac{\sinh(P-a)\omega}{\sinh P\omega}$.

This TBA is in fact quite similar to the one of the $\mathcal{N}=2$ supersymmetric $SU(P)$ Toda theory [62] (the generalization of the supersymmetric sine-Gordon model for $SU(2)$): the difference affects only which nodes correspond to massive particles, and which ones to pseudo-particles. As a result, the central charge is easily determined, $c = 2P-1$. Getting back to the particular case $P = 2n+1$, we can then fold this system to obtain (see Appendix A for the proof) the TBA for the $OSP(1/2n)$ Gross–Neveu model, whose effective central charge reads therefore $c_{\text{eff}} = 1/2(2(2n+1)-1) = 2n+1/2$.

As an example, we can discuss in more details the case of the $OSP(1/2)$ GN model whose TBA is represented in Fig. 9. We will even consider an anisotropic generalization of this TBA, where the diagram is truncated to the right in a way that is equivalent to what happened in the sigma model case.

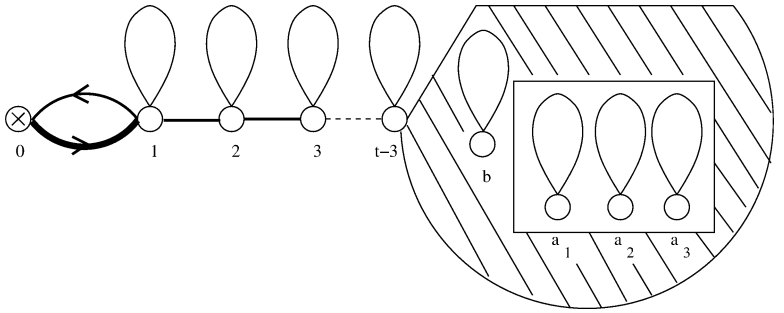

Fig. 9. Incidence diagram for the TBA describing the anisotropic version of the $OSP(1/2)$ GN model.

There is now a single color index, and we relabel $\epsilon_{a,j} = \epsilon_j$. Introducing $x_j = e^{-\epsilon_j/T}$, the equations in the UV are the same as for the $a_2^{(2)}$ Toda theory, except for the first one that reads now simply

$$x_0 = 1 + x_1. \tag{65}$$

The closure equations in particular are the same as for the $a_2^{(2)}$ anisotropic model.

The solution in the case of a vanishing twist first is

$$x_j = \frac{(j+\alpha)(j+\alpha+3)}{2},$$

in the limit $\alpha \to \infty$, for $j = 0, \ldots, t-3$. In addition one has $x_a = \frac{t-2+\alpha}{t+\alpha}$ and $x_b = \frac{(t-2+\alpha)^2}{4(t-1+\alpha)}$. As $\alpha \to \infty$, all the x's go to infinity but x_a which goes to one. As a result, the UV contribution to the central charge is $c_1 = t - 1 + 3 \times 1/2 = t + 1/2$. In the IR, the modification due to the σ_2 factor is not seen any longer, and the x's obey the same equations as for the $a_2^{(2)}$ case, with a contribution $t-2$ to the central charge. It follows that $c = 5/2$, as expected.

The twisted TBA follows from similar principles as in the sigma model case. This time however, because of $\alpha = \infty$, the UV values x_j are unaffected by the twisting. The contribution to the central charge is

$$c = t - 1 + \frac{1}{2} + (L_\lambda + L_{\lambda^{-1}})(x_a = 1) = t + \frac{1}{2} - \frac{3\alpha^2}{\pi^2}.$$

The IR values do depend on α, with formulas identical to the sigma model case, and a contribution

$$c = t - 2 - 3\frac{t-2}{t-1}\frac{\alpha^2}{\pi^2}.$$

The resulting central charge is

$$c = \frac{5}{2} - 3\frac{\alpha^2}{(t-1)\pi^2}.$$

The dependence on α is similar to what we observed in the $OSP(0,2)$ case, for similar reasons. The factor 3 in this formula, as opposed to the factor 6 in the $OSP(0,2)$ case, has

its origin in the different relations between the physical anisotropy and the parameter t in the TBA: in the $OSP(1/2)$ case, $g \approx \frac{1}{2t}$.

6. Finite field calculations

To give further evidence for our S matrices, we now present some finite field calculations. The idea, which has been worked out in great details in other cases [63], is to compare S-matrix and perturbative calculations for the ground state energy of the theory in the presence of an external field. The S-matrix calculations are considerably simpler than the TBA ones because, for a proper choice of charge coupling to the external field the ground state fills up with only one type of particles, with diagonal scattering, and the Wiener–Hopf method can be used to solve the integral equations analytically.

The S-matrix calculations are very close to the ones already performed for the $O(N)$ sigma model and the $O(N)$ Gross–Neveu models. In fact, in the region $m - 2n \geqslant 2$, the calculations are identical, since the S-matrix elements are obtained by continuation $N \equiv m - 2n$, and, in the domain $m - 2n \geqslant 2$, the integral representations are obtained by the same continuation as well. For these cases, one thus immediately checks that the continuation of the S-matrix to $N = m - 2n$ matches the beta functions of the sigma or Gross–Neveu models, which are, too, obtained by this continuation.

Things are more interesting in the case $m - 2n < 0$, in particular, to which we turn now. We consider first the OSP sigma model, with S matrices determined by σ^+. If we couple the external field to a charge of the form

$$Q \propto \int (x_1 \partial_t x_2 - x_2 \partial_t x_1)\, dx \tag{66}$$

the ground state fills up with bosonic particles of the form $|1\rangle + i|2\rangle$, with diagonal scattering $s = \sigma_2^+ + \sigma_3^+(\theta)$. If meanwhile we couple the external field to a charge of the form

$$Q \propto \int (\xi_1 \partial_t \xi_2 + \xi_2 \partial_t \xi_1)\, dx \tag{67}$$

the ground state fills up with fermionic particles of the form $|1\rangle + |2\rangle$, with diagonal scattering $s = \sigma_2^+ - \sigma_3^+$.

Let us consider this latter case. Using formulas given in the first section, one finds

$$\sigma_2^+ - \sigma_3^+ = \frac{\Gamma(1+x)}{\Gamma(1-x)} \frac{\Gamma(1/2-x)}{\Gamma(1/2+x)} \frac{\Gamma(1/2+\Delta+x)}{\Gamma(1/2+\Delta-x)} \frac{\Gamma(1+\Delta-x)}{\Gamma(1+\Delta+x)}, \tag{68}$$

where $x = \frac{i\theta}{2\pi}$ and $\Delta = \frac{1}{N-2}$. This turns out to coincide with

$$\sigma_2^- + \sigma_3^- = \frac{\Gamma(1+x)}{\Gamma(1-x)} \frac{\Gamma(1/2-x)}{\Gamma(1/2+x)} \frac{\Gamma(1/2-\Delta+x)}{\Gamma(1/2-\Delta-x)} \frac{\Gamma(1-\Delta-x)}{\Gamma(1-\Delta+x)} \tag{69}$$

after a continuation $\Delta \to -\Delta$. Similarly,

$$\sigma_2^- - \sigma_3^- = \frac{\Gamma(1+x)}{\Gamma(1-x)} \frac{\Gamma(1/2-x)}{\Gamma(1/2+x)} \frac{\Gamma(1/2-\Delta+x)}{\Gamma(1/2-\Delta-x)} \frac{\Gamma(-\Delta-x)}{\Gamma(-\Delta+x)} \tag{70}$$

does coincide with

$$\sigma_2^+ + \sigma_3^+ = \frac{\Gamma(1+x)}{\Gamma(1-x)}\frac{\Gamma(1/2-x)}{\Gamma(1/2+x)}\frac{\Gamma(1/2+\Delta+x)}{\Gamma(1/2+\Delta-x)}\frac{\Gamma(\Delta-x)}{\Gamma(\Delta+x)} \tag{71}$$

after the same continuation.

This means that, in a TBA calculation, the ground state energy of the $OSP(m/2n)$ sigma model coupled to a fermionic charge follows from known calculations about the $O(N)$ Gross–Neveu model after formally setting $N = m - 2n$ *and* performing a continuation $N-2 \to 2-N$. From expressions in [63] we find therefore, for $m-2n<0$

$$E(h) - E(0) = -\frac{h^2}{2\pi}\left[1 + \frac{1}{N-2}\frac{1}{\ln(h/m)} - \left(\frac{1}{N-2}\right)^2 \frac{\ln\ln(h/m)}{\ln^2(h/m)} + \frac{1}{N-2}\frac{C_N}{\ln^2(h/m)} + O\left(\frac{\ln\ln(h/m)}{\ln^3(h/m)}\right)\right] \tag{72}$$

with

$$C_N = \ln\Gamma\left(1+\frac{1}{N-2}\right) - \left(1-\frac{1}{N-2}\right)\ln 2 + 1,$$

and $N = m - 2n$. From this we deduce [64] the ratio of the first two coefficients of the beta function as $\beta_2/\beta_1^2 = \frac{1}{N-2}$.

Similarly, from TBA calculations, the ground state energy of the $OSP(m/2n)$ Gross–Neveu model follows from known calculations about the $O(N)$ sphere sigma model after formally setting $N = m - 2n$ *and* performing a continuation $N-2 \to 2-N$. From expressions in [63] we find therefore for $m-2n<2$

$$E(h) - E(0) = (N-2)\frac{h^2}{4\pi}\left[\ln(h/m) - \frac{1}{N-2}\ln\ln(h/m) + D_N + O\left(\frac{\ln\ln(h/m)}{\ln(h/m)}\right)\right], \tag{73}$$

where

$$D_N = -\frac{3}{N-2}\ln 2 - \left(\frac{1}{2}+\frac{1}{N-2}\right) - \ln\Gamma\left(1-\frac{1}{N-2}\right).$$

From this we deduce the ratio $\beta_2/\beta_1^2 = -\frac{1}{N-2}$. Observe that the leading term follows from the calculations for the $\beta\gamma$ system described in the previous section; all that has to be changed is the beta function for the coupling g in (59), resulting in $E_{gs} = -\frac{N-2}{4\pi}h^2\ln(h/m)$ indeed. Remarkably, it is the ground state of the Gross–Neveu model that has a leading $h^2\ln(h/m)$ dependence, while the ground state of the sigma model has a leading pure h^2 dependence: the roles of Gross–Neveu and sigma model are therefore switched compared to the usual $O(N)$ situation.

The calculation with a coupling to the first kind of charge (66) for $m-2n<2$ or the second type of charge (67) for $m-2n>2$ poses difficulties, as the kernel does not factorize in the usual way then, so the Wiener–Hopf method does not seem applicable.

Now recall that for the usual $O(N)$ sphere sigma model, $\beta_2/\beta_1^2 = \frac{1}{N-2}$, and for the usual $O(N)$ Gross–Neveu model, $\beta_2/\beta_1^2 = \frac{-1}{N-2}$. The ratios we found are thus the analytic continuations to $N \to m - 2n$, as desired.[5]

7. Conclusions and speculations

To conclude, although more verifications ought to be carried out to complete our identifications, we believe we have determined the scattering matrices for the massive regimes of the $OSP(m/2n)$ GN and the $OSP(m/2n)/OSP(m-1/2n)$ sigma models in the simple case $m = 1$, based on algebraic considerations as well as thermodynamic Bethe ansatz calculations.

It is tempting to expect that at least some of our results generalize to other cases $OSP(m/2n)$ for $m > 1$ and $m - 2n < 2$. In all these cases, we expect that the S-matrix of the sphere sigma model will be obtained from the conjecture at the beginning of this paper, with $N = m - 2n$, for $N < 2$. The S-matrix of the GN model is probably more complicated. Recall that in the case $N \geqslant 2$, it is given by the general conjecture only for $N > 4$. When $N \leqslant 2$, we think it is probably given by the conjecture only for $N < 0$.

Observe now that for the usual $O(N)$ case, the factors σ_2^+ and σ_2^- do not exhibit poles and are equal for $N = 3, 4$. For these values, the (unique) S-matrix based on the general conjecture (1) describes correctly the sigma model. As for the Gross–Neveu model, its description is more subtle: it turns out that the vector particles are actually unstable, and that the spectrum is made of kinks only.

In the case $N < 2$ of interest here, the factors σ_2^+ and σ_2^- similarly do not exhibit poles and are equal for $N = 1, 0$. These cases would correspond for instance to $OSP(3/2)$ and $OSP(2/2)$, respectively. It is very likely that there again, the S matrices describe the sigma model, and not the Gross–Neveu model, for which the proper particle content has still to be identified.

The $OSP(2/2)$ case is particularly intriguing. The $S^{1,2}$ sphere can be parameterized by

$$x_1 = \cos\phi\left(1 - \frac{1}{2}\eta_1\eta_2\right), \qquad x_2 = \sin\phi\left(1 - \frac{1}{2}\eta_1\eta_2\right),$$
$$\xi_1 = \eta_1, \qquad \xi_2 = \eta_2. \tag{74}$$

The action of the sigma model now reads

$$S = -\frac{1}{|g|}\int d^2x \left[(\partial_\mu \Phi)^2(1 - \eta_1\eta_2) + \partial_\mu\eta_1\partial_\mu\eta_2 - \eta_1\eta_2\partial_\mu\eta_1\partial_\mu\eta_2\right], \tag{75}$$

where Φ is compactified, $\Phi \equiv \Phi \bmod 2\pi$. A rescaling and a relabeling brings it into the form

$$S = \int d^2x \left[-(\partial_\mu \Phi)^2(1 + |g|\eta_1\eta_2) + \partial_\mu\eta_1\partial_\mu\eta_2 + |g|\eta_1\eta_2\partial_\mu\eta_1\partial_\mu\eta_2\right] \tag{76}$$

[5] The existence of different sectors in the $OSP(m/2n)$ models does not spoil this conclusion, as the beta functions are only trivially affected by the twists.

with now $\Phi \equiv \Phi \bmod \frac{2\pi}{\sqrt{|g|}}$. We see, in particular, that in the limit $g \to 0$, the action is simply the one of a free uncompactified boson and a free fermion, of total central charge $c = -1$, and that the boson has a *negative* coupling: this system therefore coincides with the standard "bosonization" of the $\beta\gamma$ system in the limit $|g| \to 0$!

Notice that as soon as $m > 1$, the negative sign of the bosons coupling for $OSP(m/2n)$ sigma models will have to be handled carefully; presumably the $OSP(2/2)$, $\beta\gamma$ example will provide a good example of how to do this.

Finally, we discuss the boundary value $N = 2$, which exhibits some exceptional features. Indeed, from the integrable point of view, the solution of the Yang–Baxter equation (combined with crossing and unitarity) based on the generic S-matrix is not unique for $N = 2$ (in contrast with the other values of N) but admits one continuous parameter γ. This solution is close to the sine-Gordon solution, and is related to it by

$$S = \sigma_2 + \sigma_3, \qquad S_T = \sigma_1 + \sigma_2, \qquad S_R = \sigma_1 + \sigma_3, \tag{77}$$

where S, S_T, S_R are the usual sine-Gordon amplitudes

$$S_T = -i\frac{\sinh(8\pi\theta/\gamma)}{\sin(8\pi^2/\gamma)}, \qquad S = -i\frac{\sinh(8\pi(i\pi-\theta)/\gamma)}{\sin(8\pi^2/\gamma)}S_R,$$
$$S_R = \frac{1}{\pi}\sin(8\pi^2/\gamma)U(\theta) \tag{78}$$

and $U(\theta)$ is given, e.g., in [65].

In the case of $O(2)$, the existence of this parameter corresponds to the fact that the $O(2)/O(1)$ sigma model, or the $O(2)$ GN model are actually massless critical theories, the coupling g being exactly marginal. The S-matrices then provide a massless description of these theories. Since $\sigma_2^+ = \sigma_2^-$, the $O(2)/O(1)$ sigma model and the $O(2)$ GN model coincide; their identity follows from bosonization of the massless Thirring model into the Gaussian model. The free parameter in the S-matrices is related with the coupling constant in either version of the model. (Note that the S-matrices can also be used to describe some massive perturbations. These, however, give rise to different type of models than the ones we are interested in, like the massive Thirring model.)

It seems very likely that similar things occur for $OSP(2n+2/2n)$ models as well. The identity of the sigma model and the GN model in that case is not obvious, but one can at least check using our general formulas that the central charge and the effective central charge do match, $c_{\text{eff}} = 3n + 1$, $c = 1$. There are on the other hand strong arguments showing that the beta function is exactly zero [35], so these models should have a line of fixed points indeed [10], in agreement with the S-matrix prediction.

Besides completing the identifications we have sketched here, the most pressing questions that come to mind are: what are the S-matrices of the Gross–Neveu models for non-generic values of N, what are the S-matrices for the multiflavour GN models, what are the S-matrices for the orthosymplectic Principal Chiral Models? We hope to report some answers to these questions soon. As a final related remark, recall [66] that there is an

embedding[6]

$$OSP(1/2)_{-2k} \approx SU(2)_k \times \frac{OSP(1/2)_{-2k}}{SU(2)_k} \tag{79}$$

and that the branching functions of the latter part define a Virasoro minimal model, with

$$c_{osp} = \frac{2k}{2k+3}, \qquad c_{su(2)} = \frac{3k}{k+2},$$
$$c_{\text{virasoro}} = 1 - 6\frac{(k+1)^2}{(k+2)(2k+3)}. \tag{80}$$

For k an integer, the situation is especially interesting. The Virasoro models which appear there have $p = 2k+3, q = k+2$; they are non-unitary, and their effective central charge is $c_{\text{eff}} = 1 - \frac{6}{(k+2)(2k+3)}$. These models can thus be considered as $UOSP/SU$ coset models. Their perturbation by the operator ϕ_{21} with dimension $h = 1 - \frac{3}{4(k+2)}$ coincides with the RSOS models defined in Section 3 as truncations of the $a_2^{(2)}$ theories with $t = 2k+4$. We thus see that the supersphere sigma model appears as the limit $k \to \infty$ of a series of coset models [61], just like the ordinary sphere sigma model say appears as the limit of a series of parafermion theories, this time of $SU(2)/U(1)$ type. There are many other interesting aspects of OSP coset models in relation with the present paper which we also plan to discuss elsewhere.

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Appendix A. Folding the $SU(P)$ TBAs

The quantization equations for the $SU(P)$ GN model have been written for instance in [17]:

$$2\pi P_{a0} = m_a \cosh\beta + \sum_{b=1}^{P-1} Y_{ab}^{(P)} \star \rho_{b0} - \sum_{j=1}^{\infty} \sigma_j^{(\infty)} \star \tilde{\rho}_{aj}, \quad a = 1, \ldots, P-1,$$
$$2\pi \rho_{aj} = \sigma_j^{(\infty)} \star \rho_{a0} - \sum_{b=1}^{P-1}\sum_{l=1}^{\infty} A_{jl}^{(\infty)} \star K_{ab}^{(P)} \star \tilde{\rho}_{bl}. \tag{A.1}$$

[6] The level $-2k$ in this formula stems from our conventions; it would be k if it were defined with respect to the sub $SU(2)$.

Using Fourier transform: $\hat{f}(\omega) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{Pi\omega\beta/\pi} f(\beta)$, one then has

$$\widehat{Y}_{ab}^{(P)} = \delta_{ab} - e^{|\omega|} \frac{\sinh((P-a)\omega)\sinh(b\omega)}{\sinh(P\omega)\sinh(\omega)}, \quad a \geqslant b \tag{A.2}$$

with $\widehat{Y}_{ab} = \widehat{Y}_{ba}$. The $Y^{(P)}$ kernels are logarithmic derivatives of the scattering matrix between top components in each of the fundamental representations. The group structure is encoded in the densities ρ of (massless) pseudoparticles, which appear in the solution of the auxiliary Bethe system diagonalizing the monodromy matrix.

We now set $P = 2n+1$. The folded equations then read

$$\begin{aligned} 2\pi P_{a0} &= m_a \cosh\beta + \sum_{b=1}^{n} \bigl(Y_{ab}^{(2n+1)} + Y_{a,2n+1-b}^{(2n+1)}\bigr) \star \rho_{b0} \\ &\quad - \sum_{j=1}^{\infty} \sigma_j^{(\infty)} \star \tilde{\rho}_{aj}, \quad a = 1, \ldots, n, \\ 2\pi \rho_{aj} &= \sigma_j^{(\infty)} \star \rho_{a0} - \sum_{b=1}^{n} \sum_{l=1}^{\infty} A_{jl}^{(\infty)} \star \bigl(K_{ab}^{(2n+1)} + K_{a,2n+1-b}^{(2n+1)}\bigr) \star \tilde{\rho}_{bl}. \end{aligned} \tag{A.3}$$

For instance,

$$Y_{11}^{(2n+1)} + Y_{1,2n}^{(2n+1)} = -e^{(-n+1/2)|\omega|} \frac{\sinh|\omega|}{\cosh(n+1/2)|\omega|}.$$

It is easy to check that the corresponding kernel coincides with $\sigma_3^+ - \sigma_2^+$ for $N = 1-2n$, and with the corresponding S-matrix element in the $a_{2n}^{(2)}$ scattering theory [67]. The couplings between pseudoparticles can also be checked to arise from the structure of solutions of the $a_{2n}^{(2)}$ Bethe equations, generalizing the $a_2^{(2)}$ case.

The 'sigma model like' equations for $SU(P)$ are based on a hypothetical scattering theory with physical particles in the vector representation and its conjugate only. They read

$$\begin{aligned} 2\pi P_{10} &= m\cosh\beta + Z_{11}^{(P)} \star \rho_{10} + Z_{1,P-1}^{(P)} \star \rho_{P-1,0} - \sum_{j=1}^{\infty} \sigma_j^{(\infty)} \star \tilde{\rho}_{1j}, \\ 2\pi P_{P-1,0} &= m\cosh\beta + Z_{1,P-1}^{(P)} \star \rho_{10} + Z_{P-1,P-1}^{(P)} \star \rho_{P-1,0} - \sum_{j=1}^{\infty} \sigma_j^{(\infty)} \star \tilde{\rho}_{P-1,j}, \\ 2\pi \rho_{aj} &= \sigma_j^{(\infty)} (\delta_{a1} + \delta_{a,P-1}) \star \rho_{a0} - \sum_{b=1}^{P-1} \sum_{l=1}^{\infty} A_{jl}^{(\infty)} K_{ab}^{(P)} \star\star \tilde{\rho}_{bl}, \end{aligned} \tag{A.4}$$

where

$$\widehat{Z}_{11}^{(P)} = \widehat{Z}_{P-1,P-1}^{(P)} = e^{-|\omega|} \frac{\sinh((P-1)\omega)}{\sinh(P\omega)}$$

and

$$\widehat{Z}_{1,P-1}^{(P)} = \widehat{Z}_{P-1,1}^{(P)} = e^{-|\omega|} \frac{\sinh(\omega)}{\sinh(P\omega)}.$$

The kernel $Z_{11}^{(P)}$ is the logarithmic derivative of what is called $F_{\min}^{VV}$ in [17], $\frac{1}{i}\frac{d}{d\beta}\ln F_{\min}^{VV}(\beta) = Z_{11}^{(P)}$. Setting again $P = 2n+1$ and folding gives now

$$2\pi P_{10} = m\cosh\beta + \big(Z_{11}^{(P)} + Z_{1,P-1}^{(P)}\big) \star \rho_{1,0} - \sum_{j=1}^{\infty} \sigma_j^{(\infty)} \star \tilde{\rho}_{1j},$$

$$2\pi \rho_{aj} = \sigma_j^{(\infty)} \delta_{a1} \star \rho_{a0} - \sum_{b=1}^{n}\sum_{l=1}^{\infty} A_{jl}^{(\infty)} \star \big(K_{ab}^{(2n+1)} + K_{a,2n+1-b}^{(2n+1)}\big) \star \tilde{\rho}_{bl}. \tag{A.5}$$

The kernel

$$\widehat{Z}_{11}^{(2n+1)} + \widehat{Z}_{1,2n}^{(2n+1)} = \frac{\sinh(2n\omega) + \sinh\omega}{\sinh(2n+1)\omega} e^{-|\omega|}.$$

It differs from the previous kernel

$$\widehat{Y}_{11}^{(2n+1)} + \widehat{Y}_{1,2n}^{(2n+1)} \quad \text{by} \quad -\frac{\cosh\frac{2n-3}{2}\omega}{\cosh\frac{2n+1}{2}\omega},$$

which coincides with the Fourier transform of the ratio $\frac{\sigma_2^+}{\sigma_2^-}$ for $N = 1-2n$.

Some integral representations to finish (used in the domain $N \leqslant 0$)

$$\ln\sigma_2^- = \int_0^\infty \big(e^{i(2-N)\beta\omega/\pi} + e^{-(2-N)\omega} e^{-(2-N)i\omega\beta/\pi}\big) \frac{e^{-2\omega}-1}{e^{-(2-N)\omega}+1} \frac{d\omega}{\omega} \tag{A.6}$$

and

$$\ln\frac{\sigma_2^+}{\sigma_2^-} = \int_{-\infty}^{\infty} e^{i(2-N)\beta\omega/\pi} \frac{\cosh\frac{N+2}{2}\omega}{\cosh\frac{N-2}{2}\omega} \frac{d\omega}{\omega}, \tag{A.7}$$

where

$$\sigma_2^+ = \frac{\sinh\theta - i\sin\frac{2\pi}{N-2}}{\sinh\theta + i\sin\frac{2\pi}{N-2}}\,\sigma_2^-.$$

References

[1] K.B. Efetov, Adv. Phys. 32 (1983) 53.
[2] A.W.W. Ludwig, M.P.A. Fisher, R. Shankar, G. Grinstein, Phys. Rev. B 50 (1994) 7526.
[3] D. Bernard, Conformal field theory applied to 2D disordered systems: an introduction, in: L. Baulieu, V. Kazakov, M. Picco, P. Windey (Eds.), Low-Dimensional Applications of Quantum Field Theory, Plenum, New York, 1997, pp. 19–61, hep-th/9509137.
[4] C. Mudry, C. Chamon, X.G. Wen, Nucl. Phys. B 466 (1996) 383.
[5] S. Guruswamy, A. Leclair, A.W.W. Ludwig, Nucl. Phys. B 583 (2000) 475.
[6] M. Zirnbauer, J. Math. Phys. 37 (1996) 4986.
[7] V. Gurarie, Nucl. Phys. B 546 (1999) 765.

[8] J. Cardy, Logarithmic correlations in quenched random magnets and polymers, cond-mat/9911024.
[9] M.J. Bhaseen, J.-S. Caux, I.I. Kogan, A.M. Tsvelik, Disordered Dirac fermions: the marriage of three different approaches, cond-mat/0012240.
[10] M. Bershadsky, S. Zukhov, A. Vaintrob, Nucl. Phys. B 559 (1999) 205.
[11] N. Berkovits, C. Vafa, E. Witten, JHEP 9903 (1999) 18.
[12] G. Mussardo, P. Simonetti, Phys. Lett. B 351 (1995) 515.
[13] D.C. Cabra, A. Honecker, G. Mussardo, P. Pujol, J. Phys. A 30 (1997) 8415.
[14] Z. Bassi, A. Leclair, Nucl. Phys. B 578 (2000) 577.
[15] S. Sethi, Nucl. Phys. B 430 (1994) 31.
[16] H. Saleur, B. Wehefritz-Kaufmann, Phys. Lett. B 481 (2000) 419.
[17] P. Fendley, Phys. Rev. B 63 (2001) 104429.
[18] M.J. Martins, P.B. Ramos, Nucl. Phys. B 500 (1997) 579.
[19] F. Essler, H. Frahm, preprint.
[20] M. Martins, B. Nienhuis, R. Rietman, Phys. Rev. Lett. 81 (1998) 504.
[21] I. Gruzberg, A.W.W. Ludwig, N. Read, Phys. Rev. Lett. 82 (1999) 4524.
[22] N. Read, H. Saleur, Nucl. Phys. B 613 (2001) 409.
[23] P. Fendley, H. Saleur, BPS kinks in the Gross–Neveu model, hep-th/0105148, to appear in Phys. Rev. D.
[24] A.B. Zamolodchikov, Al.B. Zamolodchikov, Ann. Phys. 120 (1979) 253.
[25] M. Karowski, H.J. Thun, Nucl. Phys. B 190 (1981) 61.
[26] A.B. Zamolodchikov, Mod. Phys. Lett. A 6 (1991) 1807.
[27] P. Martin, Potts Models and Related Problems in Statistical Mechanics, World Scientific, 1991.
[28] M. Wadati, T. Deguchi, Y. Akutsu, Phys. Rep. 180 (1987) 247, and references therein.
[29] L. Frappat, A. Sciarrino, P. Sorba, Commun. Math. Phys. 121 (1989) 457;
L. Frappat, A. Sciarrino, P. Sorba, Dictionary on Lie Algebras and Superalgebras, Academic Press, 2000, hep-th/9607161.
[30] W. Bazhanov, A.G. Shadrikov, Theor. Math. Phys. 73 (1987) 1302.
[31] See for instance, G. Delius, M. Gould, J.R. Links, Y.Z. Zhang, Int. J. Mod. Phys. A 10 (1995) 3259;
M. Gould, Y.Z. Zhang, Nucl. Phys. B 556 (2000) 529, and references therein.
[32] F.A. Smirnov, Form Factors in Completely Integrable Models of Quantum Field Theory, World Scientific, 1992.
[33] G. Takacs, G. Watts, RSOS revisited, to appear.
[34] G. Takacs, G. Watts, Nucl. Phys. B 547 (1999) 538.
[35] F. Wegner, Nucl. Phys. B 316 (1989) 663.
[36] M. Scheunert, W. Nahm, V. Rittenberg, J. Math. Phys. 18 (1977) 146.
[37] M.J. Martins, Nucl. Phys. B 450 (1995) 768;
M.J. Martins, Phys. Lett. B 359 (1995) 334.
[38] K. Sakai, Z. Tsuboi, J. Phys. Soc. Jpn. 70 (2001) 367;
K. Sakai, Z. Tsuboi, Int. J. Mod. Phys. A 15 (2000) 2329.
[39] V. Fateev, S. Lukyanov, A. Zamolodchikov, Al. Zamolodchikov, Nucl. Phys. B 516 (1998) 652.
[40] F.A. Smirnov, Int. J. Mod. Phys. A 6 (1991) 1407.
[41] N. Yu Reshetikhin, J. Phys. A 24 (1991) 2387.
[42] S.O. Warnaar, M.T. Batchelor, B. Nienhuis, J. Phys. A 25 (1992) 3077.
[43] C. Efthimiou, Nucl. Phys. B 398 (1993) 697.
[44] G. Takacs, Nucl. Phys. B 489 (1997) 532.
[45] V. Rittenberg, M. Scheunert, Commun. Math. Phys. 83 (1982) 1.
[46] H. Saleur, Nucl. Phys. B 336 (1990) 363.
[47] C. Ahn, D. Bernard, A. Leclair, Nucl. Phys. B 346 (1990) 409.
[48] A. Zamolodchikov, Nucl. Phys. B 342 (1990) 695.
[49] P. Fendley, H. Saleur, Nucl. Phys. B 388 (1992) 609.
[50] F. Ravanini, M. Staniskhov, R. Tateo, Int. J. Mod. Phys. A 11 (1996) 677.
[51] V.I. Vichirko, N.Yu. Reshetikhin, Theor. Math. Phys. 56 (1983) 805.
[52] L. Mezincescu, R.I. Nepomechie, P.K. Townsend, A.M. Tsvelik, Nucl. Phys. B 406 (1993) 681.
[53] H.G. Kausch, Nucl. Phys. B 583 (2000) 513.
[54] H. Saleur, Nucl. Phys. B 382 (1992) 486.

[55] I. Affleck, Nucl. Phys. B 257 (1985) 397.
[56] N. Read, S. Sachdev, Nucl. Phys. B 316 (1989) 609.
[57] P. Goddard, D. Olive, G. Waterson, Commun. Math. Phys. 112 (1987) 591.
[58] I.P. Ennes, A.V. Ramallo, J.M. Sanchez de Santos, $osp(1/2)$ conformal field theory, in: Trends in theoretical physics (La Plata, 1997), Conf. Proc. 419, Amer. Inst. Phys., Woodbury, NY, 1998, hep-th/9708094;
I.P. Ennes, A.V. Ramallo, J.M. Sanchez de Santos, Nucl. Phys. B 491 (1997) 574;
I.P. Ennes, A.V. Ramallo, J.M. Sanchez de Santos, Nucl. Phys. B 502 (1997) 671;
I.P. Ennes, A.V. Ramallo, J.M. Sanchez de Santos, Phys. Lett. B 389 (1996) 485.
[59] A.W.W. Ludwig, A free field representation of the $Osp(2/2)$ current algebra at level $k = -2$, and Dirac fermions in a random $SU(2)$ gauge potential, cond-mat/0012189.
[60] S. Guruswamy, A. Ludwig, Nucl. Phys. B 519 (1998) 661.
[61] P. Fendley, Phys. Rev. Lett. 83 (1999) 4468.
[62] P. Fendley, K. Intrilligator, Nucl. Phys. B 380 (1992) 265.
[63] P. Hasenfratz, F. Niedermayer, Phys. Lett. B 245 (1990) 529;
P. Forgacs, F. Niedermayer, P. Weisz, Nucl. Phys. B 367 (1991) 123.
[64] T. Hollowood, Nucl. Phys. Proc. Suppl. A 45 (1996) 130.
[65] A.B. Zamolodchikov, Al.B. Zamolodchikov, Ann. Phys. 120 (1979) 253.
[66] J. Fan, M. Yu, Modules over affine Lie superalgebras, hep-th/9304122.
[67] G.M. Gandenberger, Int. J. Mod. Phys. A 13 (1998) 4553.