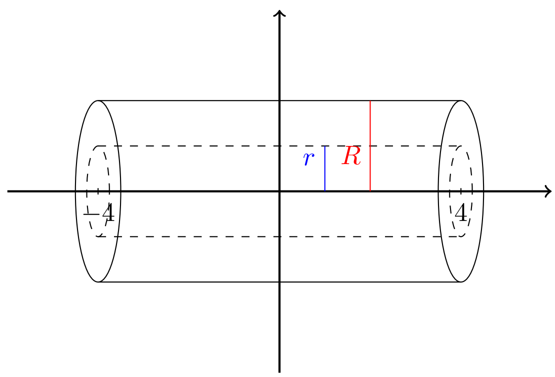
Overview

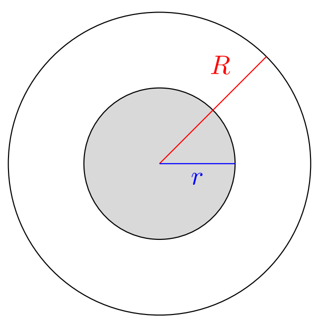
In the previous lesson we covered volumes of revolution using the disc method. Here we complicate things a bit more introducing the washer method.

Lesson

We started the previous lesson with the example of a right circular cylinder. But calculating the volume of a simple cylinder isn't very interesting. What would we do if we cut out a smaller cylinder from the middle? How would we calculate the volume of the outer shell?



Without the use of calculus, we know the volume is the area of the base times the height. The height of the cylinder is 8, and to find the area of the base, we look at a cross section.



For those that care, this shape is called an annulus. In order to compute the area, we just take the area of the whole big circle and subtract out the area of the smaller circle. That is

$$\pi R^2 - \pi r^2 = \pi(R^2 - r^2).$$

So to answer our original question of the volume of the cylindrical shell, it's just $8\pi(R^2 - r^2)$. Just as before we can translate this into an integral formula that resembles what we saw with the disc method.

Since we're just cutting out the middle of the solid, we choose dx or dy in the same way as the disc method: rotating about the y-axis is a "dy" problem and rotating about the

x-axis is a "dx" problem. So the formula to remember is

$$\text{Volume} = \pi \int_a^b \left(R^2 - r^2\right)\, dx$$

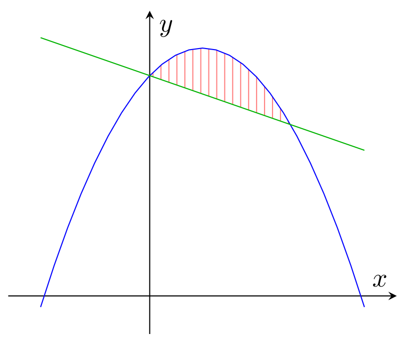
where a and b bound the region we're rotating, R is the curve *farthest* from the axis rotation and r is the curve *closest* to the axis of rotation. If we rotate about the y-axis, of course the dx becomes a dy.

Example 1. Find the volume between

$$y = -x^2 + 3x + 18 \quad \text{and} \quad y = 18 - x$$

revolved about the x-axis.

Solution. The first thing we should do is graph the desired region we want to rotate.



Since we are rotating about the x-axis, we should be thinking "dx." The parabola is farther away from the x-axis, so $R = -x^2 + 3x + 18$ and $r = 18 - x$. To determine what the limits of integration are, we need to determine where the two curves intersect.

$$\begin{aligned} -x^2 + 3x + 18 &= 18 - x \\ x^2 - 4x &= 0 \\ x(x-4) &= 0 \\ x &= 0, 4 \end{aligned}$$

Now we can put this into the formula:

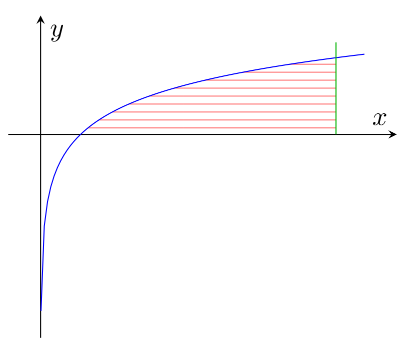
$$
\begin{aligned}
\pi \int_0^4 &\left[(-x^2+3x+18)^2 - (18-x)^2\right]\, dx \\
&= \pi \int_0^4 \left[x^4 - 6x^3 - 27x^2 + 108x + 324 - 324 + 36x - x^2\right]\, dx \\
&= \pi \int_0^4 \left(x^4 - 6x^3 - 28x^2 + 144x\right)\, dx \\
&= \pi \left(\frac{x^5}{5} - \frac{3x^4}{2} - \frac{28x^3}{3} + 72x^2\right)\bigg|_0^4 \\
&= \pi \left(\frac{4^5}{5} - \frac{3\cdot 4^4}{2} - \frac{28\cdot 4^3}{3} + 72\cdot 4^2\right) \\
&= \frac{5632}{15}\pi.
\end{aligned}
$$

Note that the only way we could have computed this integral is multiplying everything out. □

Example 2. Find the volume of the solid generated by revolving the given region about the y-axis:

$$y = \frac{1}{4}\ln x, \quad y = 0, \quad x = e^2.$$

Solution. Again we start by drawing the picture.



About the y-axis means we want "dy." So we want to have x as a function of y :

$$
\begin{aligned}
y &= \frac{1}{4}\ln x \\
4y &= \ln x \\
e^{4y} &= x.
\end{aligned}
$$

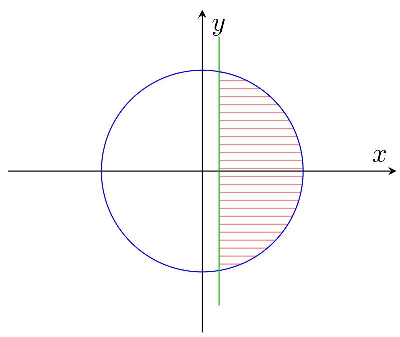
The lower bound of integration is given to be $y = 0$. The upper bound is where the two curves intersect, at $x = e^2$. When $x = e^2$, then $y = \frac{1}{4}\ln e^2 = \frac{1}{2}$. Finally, $r = e^4y$ since this is closest to the y-axis, and $R = e^2$ since this is farthest from the y-axis. Putting this all together,

$$\begin{aligned}\text{Volume} &= \pi \int_0^{1/2} \left[\left(e^2\right)^2 - \left(e^{4y}\right)^2\right]\, dy \\ &= \pi \int_0^{1/2} \left[e^4 - e^{8y}\right]\, dy \\ &= \pi \left(e^4 y - \frac{1}{8}e^{8y}\right)\Bigg|_0^{1/2} \\ &= \pi \left[\left(\frac{1}{2}e^4 - \frac{1}{8}e^4\right) + \frac{1}{8}\right] \\ &= \left(\frac{3}{8}e^4 + \frac{1}{8}\right)\pi\end{aligned}$$

□

Example 3. Find the volume of the solid generated by revolving the region inside the circle $x^2 + y^2 = 36$ and to the right of the line $x = 1$ about the y-axis.

Solution. As always, we draw a picture.



About the y-axis means "dy, and we want the right half of the circle. Solving for x, we get $x = \sqrt{36 - y^2}$; moreover, this is our R since it is farthest from the y-axis. Then our r is 1. Our limits of integration are where the vertical line and the circle intersect, which is at

$x = 1$. Using the equation of the circle, we have $y^2 + 1 = 36$, so $y = \pm\sqrt{36 - 1^2} = \pm\sqrt{35}$.

$$\begin{aligned}
\pi \int_{-\sqrt{35}}^{\sqrt{35}} \left[\left(\sqrt{36 - y^2}\right)^2 - 1^2 \right] \, dy \\
&= \pi \int_{-\sqrt{35}}^{\sqrt{35}} \left[(36 - y^2) - 1 \right] \, dy \\
&= 2\pi \int_0^{\sqrt{35}} (35 - y^2) \, dy \\
&= 2\pi \left(35y - \frac{1}{3}y^3 \right) \Big|_0^{\sqrt{35}} \\
&= 2\pi \cdot 35\sqrt{35} - \frac{1}{3} \cdot 35\sqrt{35} \\
&= \frac{4}{3}\pi \cdot 35\sqrt{35} \\
&= \frac{140}{3}\sqrt{35}\pi.
\end{aligned}$$

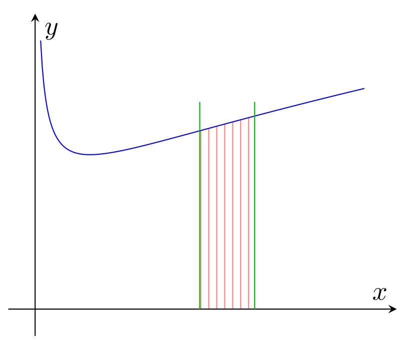
In the second equality we used the fact that $35 - y^2$ is symmetric about the y-axis. You could've skipped this step and you should still obtain the same answer. □

Example 4. Find the volume of the solid generated by rotating

$$y = \sqrt{5x} + \sqrt{\frac{x}{5}}$$

about the x-axis from $x = 3$ to $x = 4$.

Solution. Shockingly, we draw a picture.



We're explicitly given that the limits of integration are going to be $x = 3$ and $x = 4$. Since we're rotating about the x-axis, this is a "dx" problem. And even though this is the lesson on washers, it turns out that $r = 0$ since there is no bottom curve, so this is really the disc method:

$$
\begin{aligned}
\pi \int_3^4 \left(\sqrt{5x} + \sqrt{\frac{x}{5}}\right)^2\, dx \\
&= \pi \int_3^4 \left(5x + 2\sqrt{5x}\sqrt{\frac{x}{5}} + \frac{x}{5}\right)\, dx \\
&= \pi \int_3^4 \left(5x + 2\sqrt{5x \cdot \frac{x}{5}} + \frac{x}{5}\right)\, dx \\
&= \pi \int_3^4 \left(5x + 2\sqrt{x^2} + \frac{x}{5}\right)\, dx \\
&= \pi \int_3^4 \left(5x + 2x + \frac{x}{5}\right)\, dx \\
&= \pi \int_3^4 \left(7x + \frac{x}{5}\right)\, dx \\
&= \pi \left(\frac{7}{2}x^2 + \frac{1}{10}x^2\right)\Bigg|_3^4 \\
&= \pi \left(\frac{36}{10}x^2\Bigg|_3^4\right) \\
&= 25.2\pi
\end{aligned}
$$

□