Overview

In this lesson we expand on solids of revolution. In the previous two lessons we looked at the disc and washer methods about the x- and y-axes. The next natural step is to rotate about an arbitrary axis. To keep things somewhat simple, we'll stick with lines which are perpendicular to the x- or y-axes.

Lesson

The setup to these problems is largely similar to the ones from the previous two lessons. We still want to integrate

$$\int_a^b \left(R^2 - r^2\right)\, dx,$$

where R is the outer radius and r is the inner radius. The key difference here is that the value of the function is no longer the radius, so we'll have to draw a picture to determine the radius. With that, let's jump right into the examples.

Example 1. Let $\mathcal{R}$ be the region of the xy-plane bounded above by the curve $x^3y = 1728$, below by the line $y = 1$, on the left by the line $x = 6$, and on the right by the line $x = 12$. Find the volume of the solid obtained by rotating $\mathcal{R}$ around

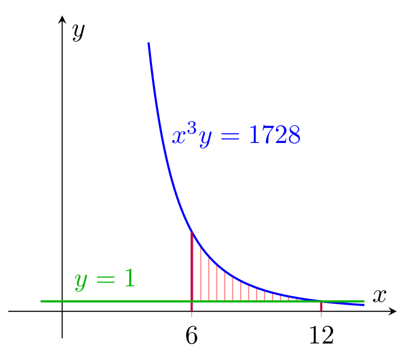
(a) the x-axis

(b) the y-axis

(c) the line $y = 1$

(d) the line $x = 6$

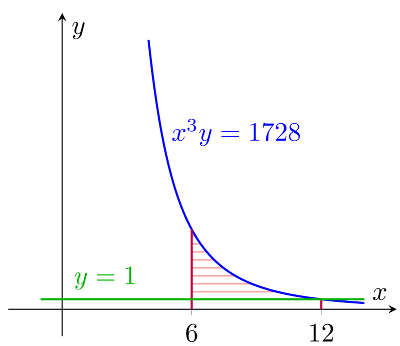
Solution. At this point, (a) and (b) should pretty straightforward, but let's work through them anyway. For (a), we want to rotate around the x-axis, so our radii are perpendicular to the x-axis, meaning we should be thinking "dx."



Then using the washer method we learned last time,

$$\begin{aligned}
\text{Volume} &= \pi \int_6^{12} \left[\left(\frac{1728}{x^3} \right)^2 - 1 \right] dx \\
&= \pi \int_6^{12} \left(1728^2 x^{-6} - 1 \right) dx \\
&= \pi \left(-\frac{1728^2}{5} x^{-5} - x \right) \Big|_6^{12} \\
&= \pi \left[\left(-\frac{1728^2}{5} \cdot \frac{1}{12^5} - 12 \right) - \left(-\frac{1728^2}{5} \cdot \frac{1}{6^5} - 6 \right) \right] \\
&= \pi \left[\left(-\frac{12}{5} - 12 \right) - \left(-\frac{384}{5} - 6 \right) \right] \\
&= \pi \left[-\frac{72}{5} + \frac{414}{5} \right] \\
&= \frac{342}{5} \pi.
\end{aligned}$$

Similarly about the y-axis:



Now we should be thinking "dy." So we need x as a function of y.

$$\begin{aligned}
x^3 y &= 1728 \\
x^3 &= \frac{1728}{y} \\
x &= \left(\frac{1728}{y} \right)^{1/3} \\
x &= 1728^{1/3} y^{-1/3} \\
&\boxed{x = 12y^{-1/3}}
\end{aligned}$$

Now the outer radius ix $x = 12y^{-1/3}$ and the inner radius is $x = 6$ for $1 \le y \le 8$. (How did we come up with 8? When $x = 6$.) Then the integral is

$$\begin{aligned}
\text{Volume} &= \pi \int_1^8 \left[\left(12y^{-1/3}\right)^2 - 6^2 \right] dx \\
&= \pi \int_1^8 \left[144y^{-2/3} - 36 \right] dx \\
&= \pi \left[432y^{1/3} - 36y \right]_1^8 \\
&= \pi \left[\left(432(8)^{1/3} - 36(8)\right) - \left(432(1)^{1/3} - 36(1)\right) \right] \\
&= \pi \left[(864 - 288) - (432 - 36) \right] \\
&= 180\pi.
\end{aligned}$$

Now how does this change when we look at parts (c) and (d)? For (c), we are now rotating about the line $y = 1$, which is one of the boundary lines for $\mathcal{R}$. This means that we will be employing the disc method (or the washer method with $r = 0$). Since $y = 1$ is parallel to the x-axis, this is still "dx." So the integral looks like

$$\pi \int_6^{12} R^2 \, dx.$$

We just need to determine R. Before, the radius was given by the y-value of the function. But now the axis of rotation is one unit closer, so $R = y - 1 = \frac{1728}{x^3} - 1$. Thus

$$\begin{aligned}
\text{Volume} &= \pi \int_6^{12} \left[\frac{1728}{x^3} - 1 \right]^2 dx \\
&= \pi \int_6^{12} \left[1728^2 x^{-6} - 2 \cdot 1728 x^{-3} + 1 \right] dx \\
&= \pi \left(-\frac{1728^2}{5} x^{-5} + 1728x^{-2} + x \right) \Bigg|_6^{12} \\
&= \pi \left(\frac{108}{5} - \frac{-114}{5} \right) \\
&= \frac{222}{5}\pi.
\end{aligned}$$

Finally about the line $x = 6$ is reminiscent of rotating about the y-axis.

$$\begin{aligned}
\text{Volume} &= \int_1^8 \left(12y^{-1/3} - 6\right)^2 dy \\
&= \pi \int_1^8 \left(144y^{-2/3} - 144y^{-1/3} + 36\right) dy \\
&= \pi \left(432y^{1/3} - 216y^{2/3} + 36y\right) \Big|_1^8 \\
&= \pi(288 - 252) \\
&= 36\pi.
\end{aligned}$$

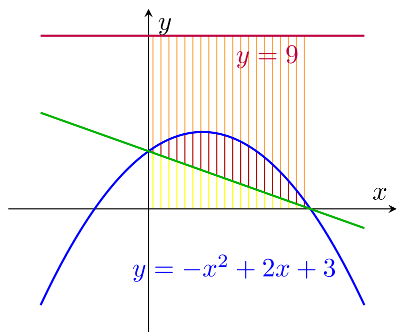
□

> **Remark.** It is important to draw a well-labeled picture for these problems to help you to determine what the radius is. Once you get that down, these problems are just the washer/disc method from there.

Example 2. Find the volume of the solid generated by revolving the given region about the line $y = 9$:

$$y = -x^2 + 2x + 3 \quad \text{and} \quad y = 3 - x.$$

Solution. Let's draw more detail for this one to help make the point clearer.



In the graph, if we think of the vertical lines as the different radii coming from the axis of rotation $y = 9$, the orange-only lines are the distance from $y = 9$ to the parabola. The orange plus the red lines are the distance from $y = 9$ to $y = 3 - x$, and all three colors give the distance from $y = 9$ to the axis. The latter is clearly a distance of 9. Now what are the yellow distances? Well they start at the x-axis and go up to the line $y = 3 - x$. So $3 - x$. This means that the length of the orange plus red lines is $9 - (3 - x) = 6 + x$. This is our outer radius. Our inner radius is the orange only part. And we obtain that in the same way. The yellow plus red distance is just the distance from the x-axis to the parabola: $-x^2 + 2x + 3$. Thus the orange only distance is $9 - (-x^2 + 2x + 3) = x^2 - 2x + 6$. And this is our inner radius. The region we are rotating is the one with the red lines, and our bounds are the points where the line and the parabola intersect. Now we have all the information we need.

$$\begin{aligned}
\text{Volume} &= \pi \int_0^3 \left[(6 + x)^2 - (x^2 - 2x + 6)^2\right] \, dx \\
&= \pi \int_0^3 \left[(36 + 12x + x^2) - (x^4 - 4x^3 + 16x^2 - 24x + 36)\right] \, dx \\
&= \pi \int_0^3 \left[36x - 15x^2 + 4x^3 - x^4\right] \, dx \\
&= \pi \left(18x^2 - 5x^3 + x^4 - \frac{1}{5}x^5\right)\Big|_0^3 \\
&= \pi \left(18(3)^2 - 5(3)^3 + (3)^4 - \frac{1}{5}(3)^5\right) \\
&= \frac{297}{5}\pi.
\end{aligned}$$

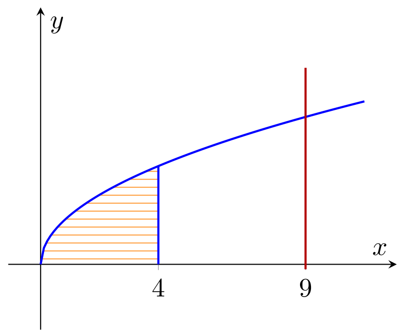
□

Example 3. Find the volume of the solid generated by revolving the region enclosed by the curves

$$y = \sqrt{x}, \quad y = 0, \quad x = 4$$

about the line $x = 9$.

Solution. As always, we draw a picture.



We're rotating about a line parallel to the y-axis. So we should be thinking "dy." Our inner radius should be easy to figure out. It's simply the distance between the lines $x = 9$ and $x = 4$, which is 5. Our outer radius is the distance from the top half of the parabola to the line $x = 9$, that is, $9 - y^2$. Since the region is bound by $y = 0$ on the bottom, that's our lower bound for the integral. And to find the upper bound, we find where $x = 4$ and $x = y^2$ intersect, which is at $x = 2$. Now

$$\begin{aligned} V &= \pi \int_0^2 \left[\left(9 - y^2\right)^2 - 5^2 \right] dy \\ &= \pi \int_0^2 \left(81 - 18y^2 + y^4 - 25\right) dy \\ &= \pi \int_0^2 \left(56 - 18y^2 + y^4\right) dy \\ &= \pi \left[56y - 6y^3 + \frac{1}{5}y^5 \right] \Big|_1^2 \\ &= \frac{352}{5}\pi. \end{aligned}$$

□

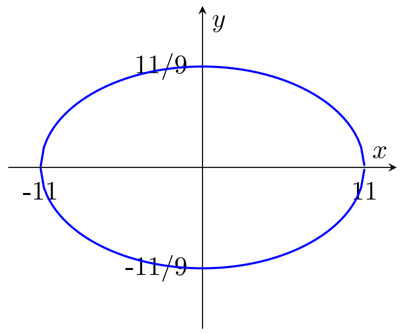
The last example should be reminiscent of the last homework problem on area between curves where we wanted to find the equation of the horizontal line which divided the area of a particular region in half.

Example 4. A propane tank is in the shape generated by revolving the region enclosed by the right half of the graph of

$$x^2 + 81y^2 = 121 \quad \text{and the } y\text{-axis}$$

about the y-axis. If x and y are measured in meters, find the depth of the propane in the tank when it is filled to one-quarter of the tank's volume.

Solution. The equation we are given is the ellipse centered at the origin with x-intercepts ± 11 and y-intercepts $\pm 11/9$.



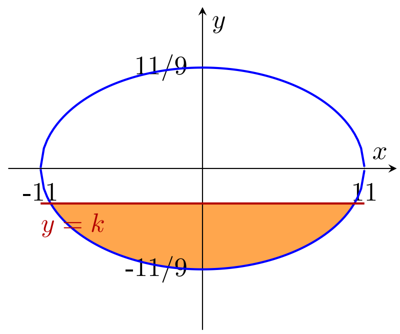
And we want to take the right half of the ellipse and rotate it about the y-axis. So first we compute the total volume of the tank. Rotating about the y-axis means "dy," so solving for x in the equation for the parabola, we get

$$x = \sqrt{121 - 81y^2}.$$

Now computing the volume of the tank,

$$\begin{aligned} V_{\text{tank}} &= \pi \int_{-11/9}^{11/9} \left(\sqrt{121 - 81y^2}\right)^2 \, dy \\ &= \pi \int_{-11/9}^{11/9} \left(121 - 81y^2\right) \, dy \\ &= 2\pi \int_{0}^{11/9} \left(121 - 81y^2\right) \, dy \\ &= 2\pi \left(121y - 27y^3\right) \Big|_0^{11/9} \\ &= \frac{5324}{27}\pi \\ &\approx 619.476. \end{aligned}$$

But the question is at what depth is $V = \frac{1}{4}V_{\text{tank}} \approx 154.869$. So just as in the previous homework problem we referenced, let $y = k$ be the y-value that achieves the desired volume.



Then revolving the orange region about the y-axis should give us a volume of 154.869. That is

$$\begin{aligned}154.869 &= \pi \int_{-11/9}^{k} \left(121 - 81y^2\right)\, dy \\ &= \pi\left(121y - 27y^3\right)\Big|_{-11/9}^{k} \\ &= \pi\left[\left(121k - 27k^3\right) - \left(121\left(-\frac{11}{9}\right) - 27\left(-\frac{11}{9}\right)^3\right)\right] \\ &= 121\pi k - 27\pi k^3 + \frac{2662}{27}\pi.\end{aligned}$$

This gives us a cubic equation, which is not easily solvable by hand. But if we plug this equation into WolframAlpha, we quickly find three solutions for k: $k = -1.87255$, $k = -.424473$, $k = 2.29703$. You'll immediately notice that the last k is not possible since $11/9 = 1.222\ldots$, and that k-value isn't inside of the ellipse. For the same reason we can't have k be -1.87255. Now remember that the question was about the *depth*, not a y-value. The depth is the distance from $y = -11/9$, and if we compute $k - (-11/9) = k + 11/9$ we get .7977. □

Remark. It is better to draw a picture and think about what the equation is for each radius than to try to memorize some algorithm. If you can label your graph, it shouldn't be too difficult.