Overview

In this lesson we discuss improper integrals. These include integrals which have infinite bounds or discontinuities at one of the bounds. This lesson also serves as a reminder that infinity is not a number and we cannot simply "plug in" ∞ into something.

Lesson

As we mentioned there are two types of improper integrals. We define Type 1 in the following way, provided that the necessary limits exist.

$$\int_a^\infty f(x)\,dx = \lim_{N\to\infty}\int_a^N f(x)\,dx \tag{1}$$

$$\int_{-\infty}^b f(x)\,dx = \lim_{N\to\infty}\int_N^b f(x)\,dx \tag{2}$$

$$\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^a f(x) + \int_a^\infty f(x)\,dx \tag{3}$$

We say that the improper integral *converges* (or is *convergent*) if the limit on the right hand side in (1), (2) and (3) exists and is finite. Otherwise we say the integral *diverges* (or is *divergent*).

Remark. Computing improper integrals involves computing limits. We don't have to venture to far into the wild before finding limits that are difficult to compute, but recall that one strategy is L'Hôpital's Rule involving indeterminate forms. Recall that indeterminate forms include $\infty - \infty, \frac{\infty}{\infty}, \frac{0}{0}, 0\cdot\infty$.

L'Hôpital's Rule states that if we have functions $f(x)$ and $g(x)$ and

$$\lim_{x\to c}\frac{f(x)}{g(x)} = \frac{\infty}{\infty} \text{ or } \frac{0}{0},$$

then if

$$\lim_{x\to c}\frac{f'(x)}{g'(x)} \tag{4}$$

exists, then the limit in (4) is equal to the original limit.

Let's look at a few examples of Type 1.

Example 1. Evaluate $\int_1^\infty \frac{1}{x}\,dx$.

Solution. Just starting from the definition (1),

$$\begin{aligned}\int_1^\infty \frac{1}{x}\,dx &= \lim_{N\to\infty}\int_1^N \frac{1}{x}\,dx \\ &= \lim_{N\to\infty} \ln x\Big|_1^N \\ &= \lim_{N\to\infty} (\ln N - \ln 1) \\ &= \lim_{N\to\infty} \ln N - \lim_{N\to\infty} \ln 1 \\ &= \lim_{N\to\infty} \ln N \\ &= \infty.\end{aligned}$$

Thus the integral diverges. □

Example 2. Evaluate $\int_1^\infty \frac{1}{x^2}\,dx$.

Solution. The setup here looks quite similar, but this one actually converges!

$$\begin{aligned}\int_1^\infty \frac{1}{x^2}\,dx &= \lim_{N\to\infty}\int_1^N x^{-2}\,dx \\ &= \lim_{N\to\infty} -\frac{1}{x}\Big|_1^N \\ &= \lim_{N\to\infty} \left(-\frac{1}{N} - (-1)\right) \\ &= \lim_{N\to\infty} -\frac{1}{N} + 1 \\ &= 1.\end{aligned}$$

□

Fun Fact. For a real number p, the integral $\int_1^\infty \frac{1}{x^p}\,dx$ converges if $p > 1$ and diverges if $p \leq 1$.

Example 3. Evaluate

$$\int_1^\infty \frac{12}{\sqrt[8]{x^3}}\,dx.$$

Solution. We could rewrite the integral as $\int_1^\infty \frac{12}{x^{3/8}}\,dx$. Then by the Fun Fact, we know right away that this integral diverges. Without knowledge of the Fun Fact though, we would proceed just as in Example 2.

$$\begin{aligned}\int_1^\infty \frac{12}{x^{3/8}}\,dx &= \lim_{N\to\infty}\int_1^N \frac{12}{x^{3/8}}\,dx \\ &= \lim_{N\to\infty} \frac{12\cdot 8}{5}x^{5/8}\Big|_1^N \\ &= \lim_{N\to\infty} \frac{96}{5}\left(x^{5/8} - 1\right) \\ &= \infty.\end{aligned}$$

□

Example 4. Evaluate
$$\int_1^\infty 5(x-1)e^{-5x}\,dx$$

Solution. We should recognize that this one requires integration by parts. Let's pick

$$u = 5(x-1) \qquad dv = e^{-5x}\,dx$$
$$du = 5 \qquad v = -\frac{1}{5}e^{-5x}$$

Then the integral is

$$\begin{aligned}
&\lim_{N\to\infty}\left[5(x-1)\cdot -\frac{1}{5}e^{-5x} - \int_1^N 5\cdot -\frac{1}{5}e^{-5x}\right]_1^N \\
&= \lim_{N\to\infty} -(x-1)e^{-5x} - \frac{1}{5}e^{-5x}\Big|_1^N \\
&= \lim_{N\to\infty} \frac{1-x}{e^{5x}} - \frac{1}{5e^{5x}}\Big|_1^N \\
&= \lim_{N\to\infty}\left(\frac{1-N}{e^{5N}} - \frac{1}{5e^{5N}}\right) - \left(0 - \frac{1}{5e^5}\right) \qquad (*)
\end{aligned}$$

To finish computing the limit in $(*)$, we need notice that we have an indeterminate form, and apply L'Hôpital's Rule:

$$\begin{aligned}
\lim_{N\to\infty}\left(\frac{1-N}{e^{5N}} - \frac{1}{5e^{5N}}\right) &= \lim_{N\to\infty} \frac{-1}{5e^{5N}} - \frac{1}{5e^{5N}} \\
&= 0 - 0 \\
&= 0.
\end{aligned}$$

Notice that we only applied L'Hôpital to the first part. The second part we could already see to be zero. Using this in $(*)$, we see that the integral is equal to $\frac{1}{5e^5}$. □

Of course calling the previous examples "Type 1" suggests that there is a Type 2. These ones are seemingly more innocent. We define them as follows. Suppose f is continuous on the interval $[a, b)$ and discontinuous at b. Then we define

$$\boxed{\int_a^b f(x)\,dx = \lim_{t\to b^-}\int_a^t f(x)\,dx,}$$

provided that the limit exists and is finite. Similarly, if f is continuous on the interval $(a, b]$ and discontinuous at a, then we define the integral

$$\boxed{\int_a^b f(x)\,dx = \lim_{t\to a^+}\int_t^b f(x)\,dx,}$$

again, provided that the limit exists and is finite.

Example 5. Evaluate
$$\int_0^1 \ln x\,dx.$$

Solution. Notice that $\ln x$ is continuous on the interval $(0,1]$ but discontinuous at 0, so this is an improper integral. Recall that we can compute the antiderivative for $\ln x$ using integration by parts with $u = \ln x$ and $dv = dx$, or simply remember it to be $x\ln x - x$ after so many times of doing it. Then

$$\begin{aligned}
\int_0^1 \ln x\,dx &= \lim_{t\to 0^+}\int_t^1 \ln x\,dx \\
&= \lim_{t\to 0^+}(x\ln x - x)\Big|_t^1 \\
&= \lim_{t\to 0^+}\left[(1\ln 1 - 1) - (t\ln t - t)\right] \\
&= \lim_{t\to 0^+}\left[-1 - (t\ln t - t)\right]. \qquad (*)
\end{aligned}$$

Now the problem is $\lim_{t\to 0^+} t\ln t$ gives $0\cdot\infty$, an indeterminate form. But we can employ L'Hôpital with the following trick

$$\begin{aligned}
\lim_{t\to 0^+} t\ln t &= \lim_{t\to 0^+}\frac{\ln t}{1/t} \\
&\overset{LH}{=} \lim_{t\to 0^+}\frac{1/t}{-1/t^2} \\
&= \lim_{t\to 0^+}(-t) = 0.
\end{aligned}$$

Now it should be easy to see that the limit in $(*)$ is simply -1. □

Example 6. Evaluate
$$\int_0^{\pi/2} \tan\theta\,d\theta.$$

Solution. Again we have a continuity problem at one of the bounds, namely $\frac{\pi}{2}$.

$$\begin{aligned}
\int_0^{\pi/2}\tan\theta\,d\theta &= \lim_{t\to\pi/2^-}\int_0^{\pi/2}\tan\theta\,d\theta \\
&= \lim_{t\to 0^+}\int_1^t -\frac{du}{u} && \begin{aligned} u &= \cos\theta \\ du &= -\sin\theta\,d\theta \end{aligned} \\
&= \lim_{t\to 0^+}\int_t^1 \frac{du}{u} \\
&= \lim_{t\to 0^+}\ln u\Big|_t^1 \\
&= \lim_{t\to 0^+}\left[\ln 1 - \ln t\right] \\
&= \infty.
\end{aligned}$$

Thus the integral diverges. When we made our u-substitution, it should be clear why we went from $t\to\pi/2$ to $t\to 0$. But why did the direction from which we were approaching change? If you think about approaching $\pi/2$ from the right along $\cos\theta$, all the y-values for $\cos\theta$ are positive. So we are approaching 0 from positive values, i.e., approaching 0 from the right. Alternatively, if this was unclear we could have immediately started with a u-substitution: $\int_0^{\pi/2} \tan\theta\, d\theta = \int_0^1 \frac{du}{u}$ and then proceed with that improper integral. $\square$

Example 7. Evaluate

$$\int_{-2}^{14} \frac{dx}{\sqrt[4]{x+2}}.$$

Solution. This one requires little more than a u-substitution to solve pretty handily ($u = x+2$ if you wish).

$$\begin{aligned}
\int_{-2}^{14} \frac{dx}{\sqrt[4]{x+2}} &= \int_{-2}^{14} (x+2)^{-1/4}\, dx \\
&= \lim_{t\to -2^+} \int_t^{14} (x+2)^{-1/4}\, dx \\
&= \lim_{t\to -2^+} \frac{4}{3}(x+2)^{3/4}\Big|_t^{14} \\
&= \lim_{t\to -2^+} \frac{4}{3}\left[(16)^{3/4} - (t+2)^{3/4}\right] \\
&= \frac{4}{3}(16)^{3/4} \\
&= \frac{32}{3}.
\end{aligned}$$

$\square$