Overview

There is no new major information in this lesson; instead we wish to apply our knowledge of geometric series to various types of word problems. These types of word problems can be difficult to parse through, so as you work through your homework, be sure to draw a picture when you can.

Lesson

Let's start with a non-word problem. Recall that a geometric series can have any starting point we wish, and can look more complicated than the simple formula we learned last time.

Example 1. Compute

$$\sum_{n=2}^{\infty} \frac{3^{0.4n}}{4^{n+1}}$$

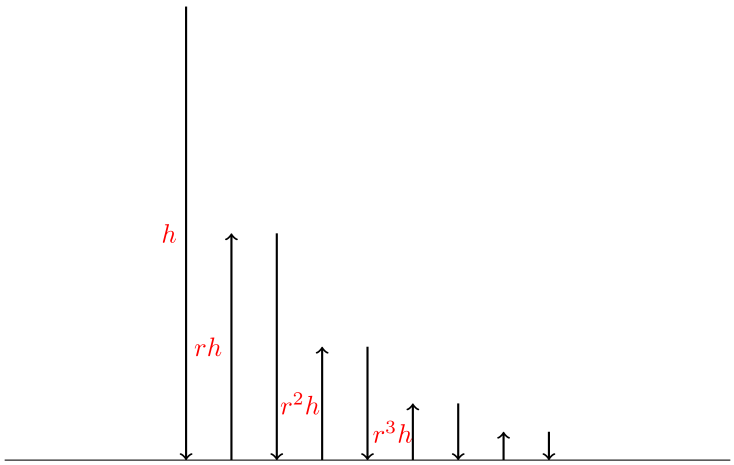
Solution. Recall the formula $\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$. So we want to get the series to look like that. If we want to start at $n = 0$, shifting the index down by 2 will increase the n's in the series by 2:

$$\begin{aligned}
\sum_{n=2}^{\infty} \frac{3^{0.4n}}{4^{n+1}} &= \sum_{n=0}^{\infty} \frac{3^{0.4(n+2)}}{4^{(n+2)+1}} \\
&= \sum_{n=0}^{\infty} \frac{3^{0.4(n+2)}}{4^{n+3}} \\
&= \sum_{n=0}^{\infty} \frac{\left(3^{0.4}\right)^{(n+2)}}{4^{n+3}} \\
&= \sum_{n=0}^{\infty} \frac{\left(3^{0.4}\right)^{2}\left(3^{0.4}\right)^{n}}{4^{3} \cdot 4^{n}} \\
&= \sum_{n=0}^{\infty} \frac{\left(3^{0.4}\right)^{2}}{4^{3}}\left(\frac{3^{0.4}}{4}\right)^{n} \\
&= \frac{\left(3^{0.4}\right)^{2}}{4^{3}} \cdot \frac{1}{1-\frac{3^{0.4}}{4}} \\
&\approx 0.06148
\end{aligned}$$

□

Example 2. A ball has the property that each time it falls from a height h onto the ground, it will rebound to a height of rh, where $r = 0.5$ and $h = 13$ meters. Find the total distance traveled by the ball.

Solution. It helps to draw a picture of the situation:



It's important to note that we're looking for *total distance traveled*. So that's

$$h + rh + rh + r^2h + r^2h + r^3h + r^3h + \cdots$$

You'll notice that all of the terms appear twice except for the initial h. So the series is *almost*

$$2\sum_{n=0}^{\infty} hr^n.$$

And if we just subtract off an h at the end, that should do the trick. So the series that describes this situation is

$$2\sum_{n=0}^{\infty} hr^n - h.$$

Using $h = 13$ and $r = \frac{1}{2}$,

$$\begin{aligned}
\text{distance traveled} &= 2\sum_{n=0}^{\infty} 13\left(\frac{1}{2}\right)^n - 13 \\
&= 2 \cdot \frac{13}{1-\frac{1}{2}} - 13 \\
&= 2 \cdot \frac{13}{\frac{1}{2}} - 13 \\
&= 2 \cdot 13 \cdot 2 - 13 \\
&= 39.
\end{aligned}$$

□

Example 3. The US discovers and colonizes an uninhabited island. Initially 552 infertile people are sent to colonize the island, and each subsequent year 552 infertile people are sent to the island. The annual death rate is 7%. Find the eventual population of the island after several years, just before a new group of 552 infertile people arrive on the island.

Solution. We want to think about how many people from each year are remaining. If the annual death rate is 7%, then 93% of the population remains. Since we are looking at the time just before a new group of people, it has been a year since the last group arrived, and there are 552(0.93) of them remaining. For the group from two years ago, there were 552(0.93) of them remaining when last year's group arrived, and another year has passed since then, so there are $552(0.93)(0.93) = 552(0.93)^2$ of that group remaining. You probably get the trend by now, so the current living population is the sum of all these:

$$552(0.93) + 552(0.93)^2 + 552(0.93)^3 + \cdots$$

And that is precisely a geometric series with $a = 552$ and $r = 0.93$. Factoring out 552(0.93),

$$\begin{aligned} &552(0.93)\big(1 + (0.93) + (0.93)^2 + \cdots\big) \\ &\quad = 552(0.93)\sum_{n=0}^{\infty}(0.93)^n \\ &\quad = 552(0.93)\cdot\frac{1}{1-0.93} \\ &\quad = 552(0.93)\cdot\frac{1}{.07} \\ &\quad = 552(0.93)\left(\frac{100}{7}\right) \\ &\quad \approx 7334. \end{aligned}$$

□

Example 4. How much money should you invest today at an annual interest rate of 6.7% compounded continuously so that, starting 2 years from today you can make annual withdrawals of $2600 in perpetuity?

Solution. A couple things to note before being able to answer this problem: "in perpetuity" is just a fancy way of saying forever, and we need to recall the continuous compounding interest formula, $A = Pe^{rt}$. Let's think about how much principal we would need each year to be able to withdraw the money we want. If we have exactly the right amount of principal, two years from now, we need to satisfy

$$2600 = P_2 e^{.067(2)},$$

where we are using P_2 to denote the principal at year 2. Solving for P_2, we get

$$P_2 = 2600e^{-.067(2)}.$$

Following the same steps, for year 3, we get

$$P_3 = 2600e^{-.067(3)}.$$

And at any year n, we need

$$P_n = 2600e^{-.067(n)}.$$

So then our initial investment should be all of these things added together to guarantee that we have enough money every year, $P = P_2 + P_3 + P_4 + \cdots$. That is,

$$\begin{aligned} P &= 2600e^{-.067(2)} + 2600e^{-.067(3)} + 2600e^{-.067(4)} + \cdots \\ &= 2600e^{-.067(2)}\left(1 + e^{-.067} + e^{-.067(2)} + \cdots\right) \\ &= 2600e^{-.067(2)} \sum_{n=0}^{\infty} e^{-.067(n)} \\ &= 2600e^{-.067(2)} \sum_{n=0}^{\infty} \left(e^{-.067}\right)^n \\ &= 2600e^{-.067(2)} \cdot \frac{1}{1-e^{-.067}} \\ &\approx 35088.98. \end{aligned}$$

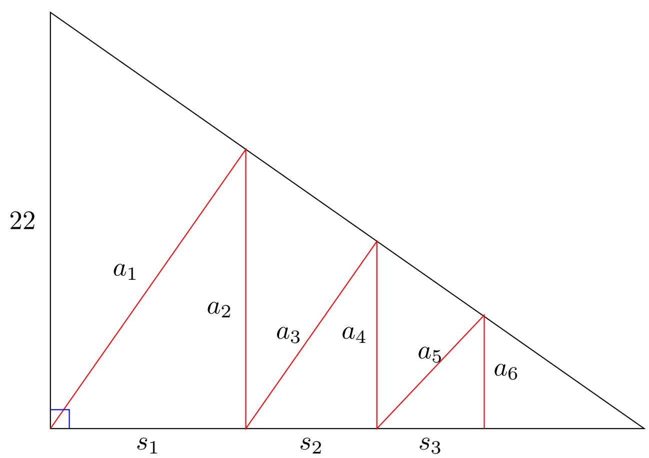
□

Example 5. A series of line segments are drawn inside a right triangle as follows:

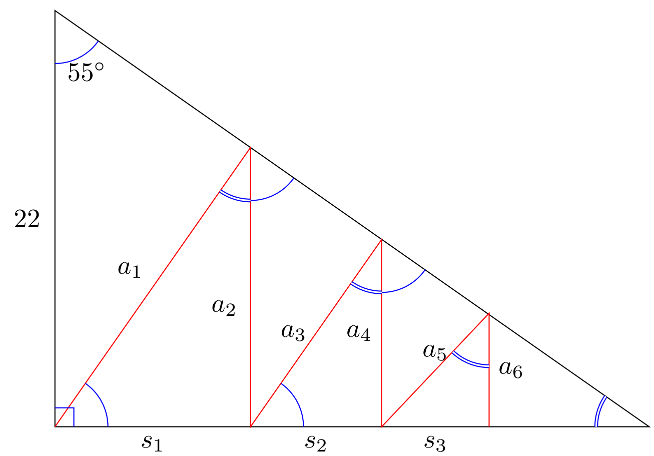
1. An altitude is drawn from the right angle of the triangle.
2. In the new smaller right triangle formed that contains the smallest angle of the original triangle, another altitude is drawn from the right angle of that triangle.
3. This process continues indefinitely, always moving toward the smallest angle of the original triangle.

Find the sum of the lengths of all these line segments if the original triangle has an angle of 55° and the side adjacent to the 55°-angle has length 22.

Solution. There's a lot going on here, so the very first thing we should do is draw a picture to illustrate the situation.



Our goal is to figure out how to express each of the a_n in terms of what we already know. From trigonometry, we know that $\sin 55° = \frac{a_1}{22}$. This immediately gives us that $a_1 = 22 \sin 55°$. To figure out a_2, it would be nice to know what one of the angles in the triangle with sides a_1, a_2, s_1 is. Using our knowledge of high school trig, we see that all of the angles marked with a single blue arc are congruent to one another, and all the angles marked with a double blue arc are congruent.



Now we can determine that $\sin 55° = \frac{a_2}{a_1}$, $\sin 55° = \frac{a_4}{a_3}, \ldots$ And solving for $a_2, a_3, \ldots,$ we see that

$$\begin{aligned} a_2 &= a_1 \sin 55° = (22 \sin 55°)(\sin 55°) = 22(\sin 55°)^2 \\ a_3 &= a_2 \sin 55° = (22 \sin 55°)^2(\sin 55°) = 22(\sin 55°)^3 \\ a_4 &= a_3 \sin 55° = (22 \sin 55°)^3(\sin 55°) = 22(\sin 55°)^4 \\ &\vdots \end{aligned}$$

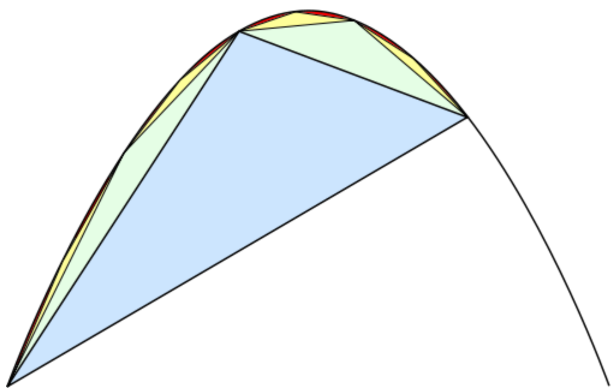
Adding up all these lengths, we notice that we have a geometric series with $r = \sin 55° \approx .89 < 1$.

$$\begin{aligned} a_1 + a_2 + \cdots &= 22 \sin 55° + 22(\sin 55°)^2 + 22(\sin 55°)^3 + \cdots \\ &= 22 \sin 55° \big(1 + \sin 55° + (\sin 55°)^2 + (\sin 55°)^3 \cdots\big) \\ &= 22 \sin 55° \sum_{n=0}^{\infty} \big(\sin 55°\big)^n \\ &= 22(\sin 55°)\frac{1}{1 - \sin 55°} \\ &\approx 99.649. \end{aligned}$$

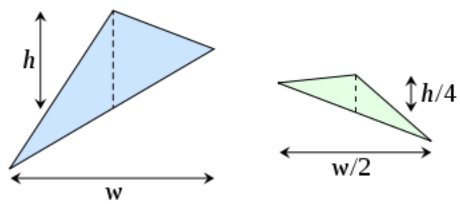
□

The next example is a fun one that dates back to the 3rd century BC.

Example 6. Find how the area of the segment of a parabola is related to the area of the triangle formed as follows: draw a chord on the parabola; the end points are two of the vertices. The third vertex is placed at the point on the parabola whose tangent line is parallel to the given chord. The triangle described is in blue below.



We make the green, yellow, red triangles and so on in the same way that we made the blue triangle. It will be helpful to know the following geometric relation between the blue triangle and each of the green triangles.



Solution. It follows from the relations above that the area of the green triangles is $\frac{1}{8}$ the area of the blue triangle. Since the yellow triangles and the red triangles are formed in the same way, we see that the area of the yellow triangles is $\frac{1}{8}$ that of the green triangles, and so on. Let A_T denote the area of the blue triangle and A denote the area of the segment of the parabola. Notice that the number number of triangles at each stage doubles. So

$$\begin{aligned} A &= A_T + 2\left(\frac{A_T}{8}\right) + 4\left(\frac{A_T}{8^2}\right) + 8\left(\frac{A_T}{8^3}\right) + \cdots \\ &= A_T + \frac{A_T}{4} + \frac{A_T}{16} + \frac{A_T}{64} + \cdots \\ &= A_T\left(1 + \frac{1}{4} + \frac{1}{4^2} + \frac{1}{4^3} + \cdots\right) \\ &= A_T \sum_{n=0}^{\infty}\left(\frac{1}{4}\right)^n \\ &= A_T \cdot \frac{1}{1-\frac{1}{4}} \\ &= \frac{4}{3}A_T. \end{aligned}$$

Thus the area of the the segment of the parabola is $\frac{4}{3}$ times the area of the described triangle. This example was proved by Archimedes in his treatise *Quadrature of the Parabola*; more information can be found at https://www2.bc.edu/mark-reeder/1103quadparab.pdf. □