Overview

In this lesson we expand on our repertoire of u-substitution problems. We discuss definite integrals with u-substitution and introduce problems where $u\,du$ and $f(x)\,dx$ differ by more than just a constant. This lesson also treats how to find the average value of a function.

Lesson

In the previous lesson after computing an antiderivative using u-substitution, we put everything back in terms of x. When we perform definite integrals, however, this is no longer necessary. How do we accomplish this? If we are given an integral

$$\int_a^b f(x)\,dx$$

and we make an appropriate substitution, we get a new integral

$$\int_{x=a}^{x=b} g(u)\,du.$$

But since we made the substitution $u =$ something in terms of x, u is a function of x. So we can plug the endpoints into this function to figure out what u is equal to at those times. Then the new integral becomes

$$\int_{u(a)}^{u(b)} g(u)\,du.$$

The difference is, we know how to calculate this integral, so we proceed as if we never made the substitution at all.

What else we might come across is that when we make our substitution, we still have some x's left over. But we shouldn't lose hope if we can solve our original substitution for x to salvage the problem. Let's look at some examples.

Example 1. Evaluate

$$\int_{-1}^{2} 12(x^2-2)(x^3-6x)^3\,dx.$$

Solution. If we pick $\boxed{u = x^3 - 6x,}$ then $du = (3x^2-6)\,dx$. This may not immediately look great, but notice that

$$\begin{aligned} du &= 3(x^2-2)\,dx \\ \frac{1}{3}\,du &= (x^2-2)\,dx. \end{aligned}$$

Now

$$\begin{aligned} \int_{-1}^{2} 12(x^2-2)(x^3-6x)^3\,dx &= 12\int_{-1}^{2} \underbrace{(x^3-6x)^3}_{u^3}\underbrace{(x^2-2)\,dx}_{\frac{1}{3}\,du} \\ &= \frac{12}{3}\int_{x=-1}^{x=2} u^3\,du = 4\int_{5}^{-4} u^3\,du \\ &= -4\int_{-4}^{5} u^3\,du = -4\cdot\frac{1}{4}u^4\Big|_{-4}^{5} \\ &= -\left((5)^4-(-4)^4\right) = 4^4-5^4 = \boxed{-369} \end{aligned}$$

□

That example helps to illuminate how we change the bounds when making a substitution, but it's not so illuminating on what we meant about having "left over x's." Hopefully the following example makes this more concrete.

Example 2. Compute

$$\int x^2\sqrt{2x-5}\,dx.$$

Solution. We start by making the u-substitution

$$\begin{aligned} u &= 2x-5 \qquad (1)\\ du &= 2\,dx \Rightarrow \frac{1}{2}\,du = dx. \end{aligned}$$

Now we have

$$\int x^2 \underbrace{\sqrt{2x-5}}_{u^{1/2}} \underbrace{dx}_{\frac{1}{2}\,du}. \qquad (2)$$

But what do we do about x^2? Using the substitution we made in (1), we can solve this for x:

$$\begin{aligned} u &= 2x-5\\ u+5 &= 2x\\ \frac{1}{2}(u+5) &= x. \end{aligned}$$

So then $x^2 = \frac{1}{4}(u+5)^2$, giving makes the integral (2)

$$\begin{aligned} &\int \frac{1}{4}(u+5)^2u^{1/2}\frac{1}{2}\,du\\ &= \frac{1}{8}\int (u+5)^2u^{1/2}\,du. \qquad (3) \end{aligned}$$

This may seem like we haven't gotten anywhere with our u-substitution, but fear not. The key difference here is we have a binomial (two terms) to an integer power and a monomial (one term) to a fractional power. So we can multiply out $(u+5)^2$ and then distribute the $u^{1/2}$.

$$\begin{aligned} (3) &= \frac{1}{8}\int (u^2+10u+25)u^{1/2}\,du\\ &= \frac{1}{8}\int (u^{5/2}+10u^{3/2}+25u^{1/2})\,du\\ &= \frac{1}{8}\left(\frac{2}{7}u^{7/2}+4u^{5/2}+\frac{50}{3}u^{3/2}\right)+C\\ &= \frac{1}{8}\left(\frac{2}{7}(2x-5)^{7/2}+4(2x-5)^{5/2}+\frac{50}{3}(2x-5)^{3/2}\right)+C. \end{aligned}$$

This may look ugly, but if we really wanted to, we could factor out $\frac{1}{21}(2x-5)^{3/2}$ and distribute the $\frac{1}{8}$ to obtain

$$\frac{1}{21}(2x-5)^{3/2}(3x^2+6x+10),$$

but we don't really care how ugly our answers look. □

Remark. The strategy of algebraically manipulating the u's and x's illustrated in Example 2 is an important one.

Example 3. Compute

$$\int \frac{5x}{\sqrt{x+6}}\,dx.$$

Solution. Making the substitution $u = x+6$, then $du = dx$ and $x = u - 6$.

$$\begin{aligned}
&\int 5x\,(x+6)^{-1/2}\,dx \\
&= 5\int (u-6)u^{-1/2}\,du \\
&= 5\int \left(u^{1/2} - 6u^{-1/2}\right)\,du \\
&= 5\left(\frac{2}{3}u^{3/2} - 12u^{1/2}\right) + C \\
&= \frac{10}{3}(x+6)^{3/2} - 60(x+5)^{1/2} + C.
\end{aligned}$$

□

Example 4. The area under the curve

$$y = te^{t^2}$$

for t in the interval $[0, x]$ is 2. What is x?

Solution. In this example we know the value of the definite integral is 2, and we are looking to find the upper bound. So we set this up as follows. Remember that x here is a number that we are solving for, not a variable like t.

$$\begin{aligned}
2 &= \int_0^x te^{t^2}\,dt \\
&= \frac{1}{2}\int_0^{x^2} e^u\,du \\
&= \frac{1}{2}e^u\Big|_0^{x^2} \\
&= \frac{1}{2}\left(e^{x^2} - e^0\right) \\
&= \frac{1}{2}\left(e^{x^2} - 1\right).
\end{aligned}$$

This gives

$$\begin{aligned}
4 &= e^{x^2} - 1 \\
5 &= e^{x^2} \\
\ln 5 &= x^2 \\
&\boxed{\sqrt{\ln 5} = x.}
\end{aligned}$$

□

The last new concept promised was that of finding the average value of a function. The discrete case is one that you are surely familiar with. If we wanted to calculate the average score on an exam, say, what would we do? We would add up the score that each student received and divided by the number of students who took the exam.

This idea extends to continuous functions. Recall that the integral $\int_a^b f(x)\,dx$ is the limit of the Riemann sum as the width of the rectangles goes to 0. Another way to think about this is that when we calculate the integral, we are really adding up every y-value that we pass through along the graph. That's a nice parallel from adding up all the exam scores. But here we can't divide by the number of x-values; there's an infinite number of them, so that doesn't make so much sense. The analogue here turns out to be the *length* of the interval $[a, b]$, which is $b - a$.

So to compute the average value of a function f on an interval $[a, b]$, we have the following formula

$$\boxed{f_{\text{avg}} = \frac{1}{b-a}\int_a^b f(x)\,dx.} \tag{4}$$

Let's see this in action.

Example 5. Find the average value of the function

$$y = \sqrt{x^2+1}x^3$$

for the interval $1 \leq x \leq 3$.

Solution. Using the formula (4), we have

$$\frac{1}{3-1}\int_1^3 \sqrt{x^2+1}x^3\,dx. \tag{5}$$

Let's make the substitution $\boxed{u = x^2+1,}$ so that $du = 2x\,dx$. Then (5) becomes

$$\begin{aligned}
&\frac{1}{2}\int_1^3 \underbrace{\sqrt{x^2+1}}_{u^{1/2}}\,x^2\,\underbrace{x\,dx}_{\frac{1}{2}\,du} \\
&= \frac{1}{4}\int_{x=1}^{x=3} u^{1/2}x^2\,du.
\end{aligned}$$

But using our original substitution, we see that $x^2 = u - 1$, and $u = 2$ when $x = 1$, $u = 10$ when $x = 1$ to give us

$$\begin{aligned}
&\frac{1}{4}\int_2^{10} u^{1/2}(u-1)\,du \\
&= \frac{1}{4}\int_2^{10} \left(u^{3/2} - u^{1/2}\right)\,du \\
&= \frac{1}{4}\left(\frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2}\right)\Bigg|_2^{10} \\
&= \frac{1}{4}\left(\frac{2}{5}(10)^{5/2} - \frac{2}{3}(10)^{3/2}\right) - \left(\frac{2}{5}(2)^{5/2} - \frac{2}{3}(2)^{3/2}\right)
\end{aligned}$$ □

Finally, no lesson is truly complete without a couple of good word problems.

Example 6. You decide to build a sandcastle in your driveway. Since your driveway is concrete and nowhere near a beach, you purchase a large 100 L bag of sand. Since the bag is very heavy, you choose to drag the bag to carry it up the driveway creating a growing hole that allows sand to escape at a changing rate of

$$s'(t) = 5 - t^2(t^3+3)^2 \quad \text{liters per minute.}$$

How much sand escaped during the time interval between 5 and 30 seconds after the bag ripped?

Solution. Notice that the rate we are given is in liters per *minute*, but the time interval is given in *seconds*. To make up for this note that 5 and 30 seconds are $\frac{1}{12}$ and $\frac{1}{2}$ minutes, respectively.

$$\begin{aligned}
&\int_{1/12}^{1/2} \left(5 - t^2(t^3+3)^2\right)\, dt \\
&= \int_{1/12}^{1/2} 5\, dt - \frac{1}{3}\int_{t=1/12}^{t=1/2} u^2\, du && u = t^3+3 \\
&= \int_{1/12}^{1/2} 5\, dt - \frac{1}{3}\int_{5185/1728}^{25/8} u^2\, du && du = 3t^2 dt \\
&= 5t|_{1/12}^{1/2} - \frac{1}{9}u^3\Big|_{5185/1728}^{25/8} \\
&= 5\left(\frac{1}{2} - \frac{1}{12}\right) - \frac{1}{9}\left[\left(\frac{25}{8}\right)^3 - \left(\frac{5185}{1728}\right)^3\right] \\
&\approx 1.69 \quad \text{liters.}
\end{aligned}$$

□

Example 7. The optimal brewing temperature for coffee is about 200°F. If the coffee doesn't sit on a hotplate, it has been observed that the temperature falls at a rate of

$$T'(t) = -5e^{-.05t} \quad {}^\circ\text{F/min.}$$

What is the average temperature of the coffee during the first 15 minutes after being brewed?

Solution. Since we want the average *temperature* for the first 15 minutes, the first thing we want to find is a function for the temperature with respect to time. Notice that we are given a function which describes the *rate of change* of the temperature function. So

$$\begin{aligned}
T(t) = \int -5e^{-.05t}\, dt &= -5\int e^{-.05t}\, dt && (u = -.05t) \\
&= \frac{-5}{-.05}\int e^u\, du && (du = -.05\, dt) \\
&= 100e^u + C
\end{aligned}$$

So $T(t) = 100e^u + C$. Using that $T(0) = 200 = 100e^0 + C$, we have that $C = 100$. Thus

$$T(t) = 100e^{-.05t} + 100.$$

Now to find the average temperature over the first 15 minutes, we calculate

$$\begin{aligned}
T_{\text{avg}} &= \frac{1}{15}\int_0^{15} (100e^{-.05t} + 100)\, dt \\
&= \frac{1}{15}\int_0^{15} 100e^{-.05t}\, dt + \frac{1}{15}\int_0^{15} 100\, dt \\
&= \frac{1}{-.05}\frac{100}{15}\int_{t=0}^{t=15} e^u\, du + \frac{1}{15}\int_0^{15} 100\, dt && (u = -.05t) \\
&= -\frac{400}{3}\int_0^{-.75} e^u\, du + \frac{1}{15}\int_0^{15} 100\, dt && (du = -.05dt) \\
&= \frac{400}{3}\int_{-.75}^{0} e^u\, du + \frac{1}{15}\int_0^{15} 100\, dt \\
&= \frac{400}{3}\, e^u\Big|_{-.75}^{0} + \frac{1}{15}\left(100t\Big|_0^{15}\right) \\
&= \frac{400}{3}\left(e^0 - e^{-.75}\right) + \frac{1}{15}(100 \cdot 15 - 0) \\
&= \frac{400}{3}\left(1 - e^{-.75}\right) + 100 \\
&\approx 170.35.
\end{aligned}$$

So the average temperature over the first 15 minutes is about 175°F. □

Remark. In the previous example it is tempting to just use the average value formula (4) and call it a day. But if you take another look at the formula, you'll notice that in order to find the average value of a function f, we end up integrating the same function f. This is an important distinction that is sure to show up again.