Overview

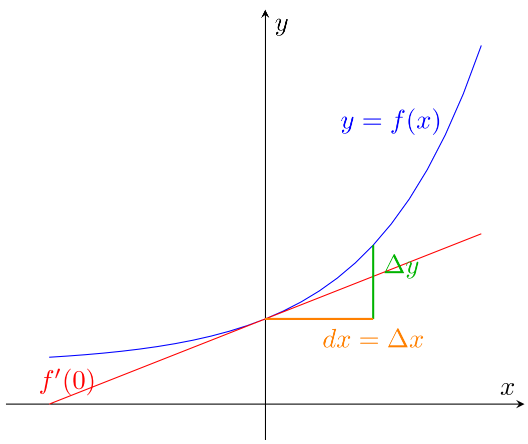
One of the main points of calculus is to take something ugly and approximate it with something that's a bit nicer to work with. In first semester calculus you learn that we can use the derivative of a function approximate its values on a small interval. This turns something that's not linear into something linear, which is far easier on calculations. One way of thinking about this situation is using the notion of the differential, and in this lesson we extend that to the total differential for two variables.

Lesson

Let's start with the one-variable situation. If we have a function $y = f(x)$ and we consider a small change in x we can define the differential $\mathrm{d}x = \Delta x$. Then this gives rise to the differential of y, which is defined as

$$\mathrm{d}y = f'(x)\,\mathrm{d}x.$$

So $\mathrm{d}y$ is the change in the height of the tangent line for the given change in x. This can be used to approximate the change in the height of the function, denoted Δy. The picture below describes this situation, where we're considering the point $x = 0$.



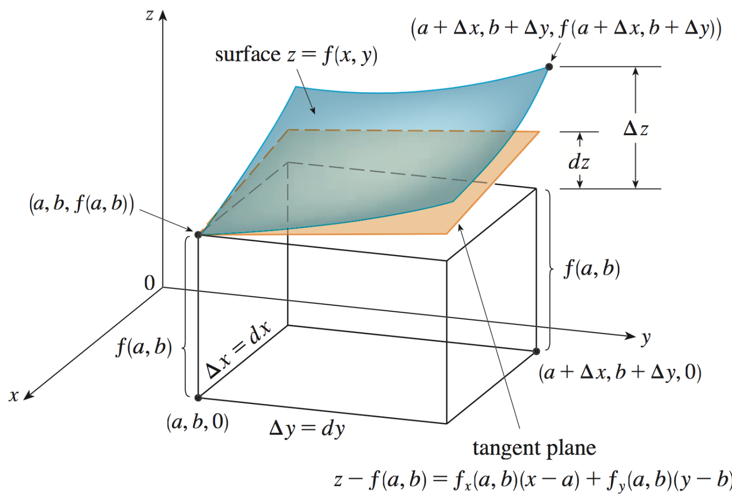
We can extend this to two variables using partial derivatives. Now we think of $\mathrm{d}x = \Delta x$ and $\mathrm{d}y = \Delta y$ being independent variables which we are allowed to choose, and $z = f(x, y)$. Then the differential of z, which we call the *total differential*, is given by

$$\mathrm{d}z = f_x(x, y)\,\mathrm{d}x + f_y(x, y)\,\mathrm{d}y \tag{1}$$

$$\mathrm{d}z = \frac{\partial z}{\partial x}\,\mathrm{d}x + \frac{\partial z}{\partial y}\,\mathrm{d}y \tag{2}$$

With this extension, $\mathrm{d}z$ describes the change in the height of the tangent plane, whereas Δz describes the change in height of the surface $z = f(x, y)$ for the given changes in x and y.

This picture is a bit harder to draw, but here's a graphic from James Stewart's (not the actor) *Calculus*.



With this understanding, we should be able to answer any of the questions that come up in the homework. So Let's go through some examples.

Example 1. Estimate the change in z at the point $(6, 8)$ for a change in x of 0.4 and a change in y of 0.6 given

$$\frac{\partial z}{\partial x} = -7x - 6 \quad \text{and} \quad \frac{\partial z}{\partial y} = 3y + 10.$$

Solution. Using (2), we have

$$\mathrm{d}z = (-7x - 6)\,\mathrm{d}x + (3y + 10)\,\mathrm{d}x,$$

and using the point $(6, 8)$, $\Delta x = \mathrm{d}x = 0.4$ and $\Delta y = \mathrm{d}y = 0.6$, we get

$$\begin{aligned} \mathrm{d}z &= (-7(6) - 6)\,(0.4) + (3(8) + 10)\,(0.6) \\ &= (-48)(0.4) + (34)(0.6) \\ &= 1.2. \end{aligned}$$

□

Example 2. Approximate to 3 decimal places

$$\sqrt{(7.5)^2 + (8.5)^2} - \sqrt{7^2 + 8^2}.$$

Solution. Of course we could just plug this into our calculator. (Let's call that the "preferred" method.) But why do that when we can use the total differential?!

If we make our function $f(x, y) = \sqrt{x^2 + y^2}$ we can use $x = 7$, $y = 8$ and $\mathrm{d}x = \mathrm{d}y = 0.5$ to approximate this value. Then we could rewrite that equation as

$$\Delta z = f(7.5, 8.5) - f(7, 8)$$

Then using (1) to approximate Δz,

$$\begin{aligned}
\mathrm{d}z &= \left(\frac{x}{\sqrt{x^2+y^2}}\right)\mathrm{d}x + \left(\frac{y}{\sqrt{x^2+y^2}}\right)\mathrm{d}y \\
&= \left(\frac{7}{\sqrt{7^2+8^2}}\right)(0.5) + \left(\frac{8}{\sqrt{7^2+8^2}}\right)(0.5) \\
&= \frac{7.5}{\sqrt{7^2+8^2}} \\
&= \frac{7.5}{\sqrt{113}} \\
&\approx 0.706.
\end{aligned}$$

Of course in this last step we had to use a calculator, so perhaps the preferred method was a better use of our time. □

Remark. After this example you might start to feel like this lesson seems pretty pointless. And you wouldn't be wrong, but it does allow us to answer some interesting word problems. The key will just be figuring out what corresponds to each of $\partial z/\partial x$, $\partial z/\partial y$, $\mathrm{d}x$ and $\mathrm{d}y$. After that, it's just plugging things in to (2).

Example 3. The pressure of an ideal gas, measured in kPa, is related to its volume V and temperature T by the equation

$$PV = 0.34T.$$

The temperature is measured with an error of 3 Kelvin and the volume is measured with an error of 0.8m^3. If it is known that the actual values are $T = 244$ K and $V = 3\text{m}^3$, what is the estimated maximum error in the measurement of pressure?

Solution. Since we want to estimate the error in the calculation of pressure, P, we should write P as a function of T and V:

$$P = \frac{0.34T}{V}.$$

Here we have $T = 244$, $V = 3$ and $\Delta T = \mathrm{d}T = \pm 3$, $\Delta V = \mathrm{d}V = \pm 0.8$. We need to include the $\pm$ because we're talking about range of error. Then using these values and (2),

$$\begin{aligned}
\mathrm{d}P &= \frac{\partial P}{\partial T}\,\mathrm{d}T + \frac{\partial P}{\partial V}\,\mathrm{d}V \\
&= \frac{0.34}{V}\,\mathrm{d}T - \frac{0.34T}{V^2}\,\mathrm{d}V \\
&= \frac{0.34}{3}(\pm 3) - \frac{0.34(244)}{3^2}(\pm 0.8).
\end{aligned}$$

Notice that this gives us 4 answers depending on the signs of 3 and 0.8 that we choose. If we plug in all the possible combinations of ± 3 and ± 0.8 and pick the largest number, that will correspond to our maximum error.

$$\frac{0.34}{3}(3) - \frac{0.34(244)}{3^2}(0.8) = -7.034 \tag{3}$$
$$\frac{0.34}{3}(-3) - \frac{0.34(244)}{3^2}(-0.8) = 7.034 \tag{4}$$
$$\frac{0.34}{3}(3) - \frac{0.34(244)}{3^2}(-0.8) = 7.714 \tag{5}$$
$$\frac{0.34}{3}(-3) - \frac{0.34(244)}{3^2}(0.8) = -7.714. \tag{6}$$

Thus the maximum error is ± 7.714. Notice that we really only needed to calculate (3) and (5) since (3) and (4) and (5) and (6) are each negatives of one another. □

Example 4. A soft drink can is a cylinder H cm tall with radius r cm. Its volume is given by the formula $V(r, h) = \pi r^2 h$. A particular can is 10 cm tall and has a radius of 3 cm. If the height is increased by 1.4 cm, estimate the change in the radius needed so that the volume stays the same.

Solution. In this example, since we want to keep V constant that means we want $\Delta V = 0 = \mathrm{d}V$. Also here we have $\Delta h = \mathrm{d}h = 1.4$, $h = 10$, $r = 3$ and we are looking for $\mathrm{d}r$. With this in mind, we use (1) to get

$$\begin{aligned}
\mathrm{d}V &= V_r(r,h)\,\mathrm{d}r + V_h(r,h)\,\mathrm{d}h \\
\mathrm{d}V &= 2\pi rh\,\mathrm{d}r + \pi r^2\,\mathrm{d}h \\
0 &= 2\pi(3)(10)\,\mathrm{d}r + \pi(3)^2(1.4) \\
0 &= 60\pi\,\mathrm{d}r + 12.6\pi \\
\mathrm{d}r &= -\frac{12.6\cancel{\pi}}{60\cancel{\pi}} \\
\mathrm{d}r &= -0.21.
\end{aligned}$$

So if we increase the height by 1.4 cm, we would need to *decrease* the radius by 0.21 cm to keep the volume the same. □

Example 5. The specific gravity of an object with density greater than that of water can be determined by using the formula

$$S = \frac{A}{A - W},$$

where A and W are the wights of the object in air and water, respectively. If the measurements of an object are $A = 2$ pounds and $W = 1.7$ pounds with maximum errors of 0.03 pounds and 0.08 pounds, respectively, find the approximate relative percentage error in calculating S.

Solution. Here we're given $A = 2$, $W = 1.7$, $\mathrm{d}A = \pm 0.03$ and $\mathrm{d}W = \pm 0.08$. Calculating the total differential $\mathrm{d}S$,

$$\begin{aligned}\mathrm{d}S &= \frac{\partial S}{\partial A}\,\mathrm{d}A + \frac{\partial S}{\partial W}\,\mathrm{d}W \\ &= \frac{(A-W)-A}{(A-W)^2}\,\mathrm{d}A + \frac{A}{(A-W)^2}\,\mathrm{d}W \\ &= \frac{-W}{(A-W)^2}\,\mathrm{d}A + \frac{A}{(A-W)^2}\,\mathrm{d}W \\ &= \frac{-1.7}{(2-1.7)^2}(\pm 0.03) + \frac{2}{(2-1.7)^2}(\pm 0.08).\end{aligned}$$

Calculating these numbers, we get ± 1.21 and $\pm 2.3\bar{4}$. So the maximum error in calculating S is $\pm 2.3\bar{4}$. To determine the relative percentage error, we calculate

$$\frac{\mathrm{d}S}{S} = \frac{2.3\bar{4}}{2/(2-1.7)^2} = .3517,$$

thus the relative error is 35.17%. □