Overview

Sometimes we have functions of multiple variables whose arguments are in turn actually functions of yet another variable. The multivariate version of the chain rule allows us to compute the rates of change of these types of functions.

Lesson

Recall the chain rule for one-variable calculus: Consider the composition of functions $h(x) = (f \circ g)(x)$. If g is differentiable at x and f is differentiable at $g(x)$, then h is differentiable and

$$\boxed{h'(x) = f'(g(x))g'(x)}$$

In Leibniz notation, if we say $u = g(x)$ and $y = f(u)$, then

$$\boxed{\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}u}\frac{\mathrm{d}u}{\mathrm{d}x}} \tag{1}$$

We can extend the chain rule to several variables, and the result should remind us of the total differential from the previous lesson and the chain rule for one variable as in (1). For two variables, if $z = f(x, y)$ is a differentiable function of x and y, and $x = g(t)$, $y = h(t)$ are differentiable functions of t, then z is a differentiable function of t, and

$$\boxed{\frac{\mathrm{d}z}{\mathrm{d}t} = \frac{\partial z}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial z}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t}} \tag{2}$$

The key to the homework for this lesson is mainly going to be knowing and using the formula in (2) and remembering how to compute partial derivatives and regular derivatives.

Example 1. Find $\frac{\mathrm{d}z}{\mathrm{d}t}$, where $z = \sin(x^2 + y^2)$, $x = 8t^2 + 3$ and $y = 7t^3$.

Solution. Using (2),

$$\begin{aligned} \frac{\mathrm{d}z}{\mathrm{d}t} &= \frac{\partial z}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial z}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t} \\ &= \left[2x\cos(x^2+y^2)\right](16t) + \left[2y\cos(x^2+y^2)\right]\left(21t^2\right) \\ &= 32tx\cos(x^2+y^2) + 22ty\cos(x^2+y^2) \end{aligned}$$

□

Remark. When computing these derivatives, it's fine to leave everything in terms of x, y and t. In fact, in Loncappa it is necessary to do so.

Example 2. Compute $\frac{\mathrm{d}z}{\mathrm{d}t}$ of

$$z = \frac{4x}{y}$$

where

$$x = e^{-4t} \quad \text{and} \quad y = 4t^2$$

at $t = 1$.

Solution.

$$\begin{aligned}\frac{\mathrm{d}z}{\mathrm{d}t} &= \frac{\partial z}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial z}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t} \\ &= \left(\frac{4}{y}\right)\left(-4e^{-4t}\right) + \left(-\frac{4x}{y^2}\right)(8t),\end{aligned}$$

When $t = 1$, then $x = e^{-4}$ and $y = 4$. Plugging these values in,

$$\frac{4}{4}\left(-4e^{-4}\right) + \left(-\frac{4e^{-4}}{4^2}\right)(8) = -\frac{6}{e^4}.$$

□

Example 3. The daily revenue from clothing sales at your favorite retailer is given by

$$R(a, w) = 10 + 6a^{3/2}w^{7/3},$$

where a dollars are spent daily on advertising and w dollars are spent daily on employee wages. It is determined that t days from now,

$$a = t^2 + t - 3 \quad \text{and} \quad w = \sqrt{t} - 1.$$

At what rate will the daily revenue be changing 4 days from now?

Solution. The takeaway here is we still have a function $R(a, w)$ and a and w are functions of t, and we are looking for $\frac{\mathrm{d}R}{\mathrm{d}t}$ at $t = 4$.

$$\frac{\mathrm{d}R}{\mathrm{d}t} = \left(6 \cdot \frac{3}{2}a^{1/2}w^{7/3}\right)(2t + 1) + 6a^{3/2}\left(\frac{7}{3}w^{4/3}\right)\left(\frac{1}{2}t^{-1/2}\right)$$

When $t = 4$ we have $a = 4^2 + 4 - 3 = 17$ and $w = \sqrt{4} - 1 = 1$, and evaluating at this points,

$$\begin{aligned}\left.\frac{\mathrm{d}R}{\mathrm{d}t}\right|_{t=4} &= 9\sqrt{17} \cdot 9 + 6 \cdot 17^{3/2} \cdot \frac{7}{3} \cdot \frac{1}{4} \\ &\approx \$579.30 \text{ per day}\end{aligned}$$

□

The next example should be reminiscent of your first semester calculus days and related rates. But with the chain rule in two variables, this becomes much easier.

Example 4. The radius of a right circular cylinder is increasing at a rate of 8 in/min and the height is decreasing at a rate of 13 in/min. What is the rate of change of the surface area when the radius is 17.5 in and the height is 29 in?

Solution. So here we have $\frac{\mathrm{d}r}{\mathrm{d}t} = 8$ and $\frac{\mathrm{d}h}{\mathrm{d}t} = -13$, and we are looking at the time $r = 17.5$ and $h = 29$. Recall that the surface area of a right circular cylinder is $S = 2\pi rh + 2\pi r^2$. So

$$\begin{aligned}\frac{\mathrm{d}S}{\mathrm{d}t} &= \frac{\partial S}{\partial r}\frac{\mathrm{d}r}{\mathrm{d}t} + \frac{\partial S}{\partial h}\frac{\mathrm{d}h}{\mathrm{d}t} \\ &= (2\pi h + 4\pi r)\frac{\mathrm{d}r}{\mathrm{d}t} + (2\pi r)\frac{\mathrm{d}h}{\mathrm{d}t} \\ &= [2\pi(29) + 4\pi(17.5)](8) + [2\pi(17.5)](-13) \\ &\approx 1787.6 \text{ in}^2/\text{min}\end{aligned}$$

□

Example 5. The monthly demand for a Donald Trump Chia Pet is given by

$$D(x, y) = \frac{1}{200} x e^{\frac{xy}{1000}} \text{ Chia Pets,}$$

where x dollars are spent on infomercials and y dollars are spent on door-to-door sales. If t months from now $x = 80 + t^{2/3}$ dollars are spent on infomercials and $y = \ln(1 + t)$ dollars are spent on door-to-door sales, at approximately what rate will the demand be changing with respect to time 8 months from now?

Solution. The only thing that makes this example a little different is we need to pay a little more attention when computing $\frac{\partial D}{\partial x}$ since this involves the product rule.

$$\begin{aligned}
\frac{\mathrm{d}D}{\mathrm{d}t} &= \frac{\partial D}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial D}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t} \\
&= \left[\frac{1}{200} x \cdot \frac{y}{1000} e^{\frac{xy}{1000}} + \frac{1}{200} e^{\frac{xy}{1000}}\right]\left(\frac{2}{3} t^{-1/3}\right) \\
&\qquad + \left[\frac{1}{200} \cdot \frac{x^2}{1000} e^{\frac{xy}{1000}}\right]\left(\frac{t}{1+t} + \ln(1+t)\right)
\end{aligned}$$

Then when $t = 8$, we have $x = 84$ and $y = 8 \ln 9$. Plugging in these values into what we've computed above leads to about 0.495 Chia Pets per month. $\square$