Overview

In one-variable calculus you learned that if a function has a local maximum or minimum, then the derivative is zero. Such a point with zero derivative is called a critical point. The first method you learned to classify critical points was the so called *first derivative test*. Later you learned the *second derivative test* which was much quicker when the test didn't fail. In this lesson we generalize the second derivative test for functions of two variables.

Lesson

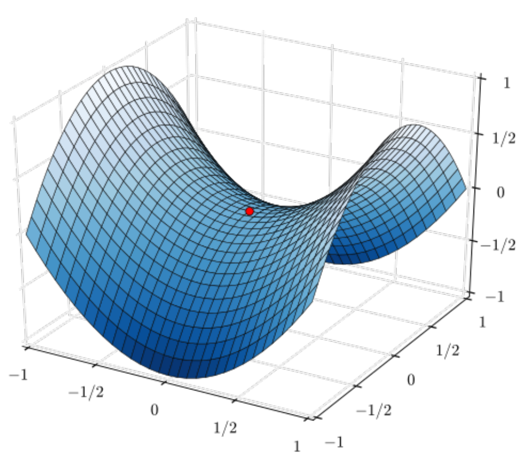
Just as in functions of a single variable the definition of a local extremum is unsurprising.

> **Definition.** A function $f(x, y)$ has a *local maximum* (*minimum*) at the point (a, b) if we have $f(x, y) \leq f(a, b)$ $(f(x, y) \geq f(a, b))$ for all points (x, y) near (a, b). We say that the maximum (minimum) occurs at (a, b) and $f(a, b)$ is the maximum (minimum) *value*.

Note. The terms local extrema and relative extrema are used interchangeably.

Moreover, as in one-variable calculus, if $f(x, y) \leq f(a, b)$ for *every* (x, y) in the domain of f, then $f(a, b)$ is an *absolute maximum* of f. Similarly we have an *absolute minimum* if $f(x, y) \geq f(a, b)$ for every (x, y) in the domain of f.

The question becomes how do we find extrema of a function of two variables? We can't just "set the derivative equal to zero" as in Calculus I, but it turns out that if f does have a local extremum at the point (a, b), then $f_x(a, b) = 0$ and $f_y(a, b) = 0$. Such a point is called a *critical point*. The first derivative test is no longer feasible, for there could be some directions where the $f(a, b)$ is a local maximum, but traveling along a different direction $f(a, b)$ is a local minimum. Such a point is called a saddle point and is illustrated below.



The graph above is that of $z = x^2 - y^2$. (Image courtesy of Wikipedia.) The red dot at the origin is a saddle point. Notice as you walk along the x direction the surface is curved upward at that point, but along the y direction, the surface is curved downward.

The picture should make it clear why this is called a saddle point. A more tangible example is that of a Pringles chip.

Since we don't have a first-derivative test, we look to the second-derivative test. As it turns out, there is an analogue for functions of two variables. If you look up the second derivative test elsewhere, you'll see that it's the determinant of the matrix of the second partial derivatives (we'll talk about matrices later), and that's a good way to remember the formula. But in this class it will be given to you on your formula sheet. Without further adieu,

The Second Derivative Test (SDT)**.** Suppose (a, b) is a critical point of f and the second partial derivatives exist and are continuous. Let

$$D = D(a, b) = f_{xx}(a, b)f_{yy}(a, b) - [f_{xy}(a, b)]^2 .$$

Then

(a) If $D > 0$ and $f_{xx}(a, b) > 0$, then $f(a, b)$ is a local minimum.

(b) If $D > 0$ and $f_{xx}(a, b) < 0$, then $f(a, b)$ is a local maximum.

(c) If $D < 0$, then $f(a, b)$ is a saddle point.

(d) If $D = 0$ then the test is inconclusive.

With this test in mind, we aim to classify the critical points in the following examples.

Example 1. Find all local extrema of

$$f(x, y) = 11 + 8x - 5y - 3x^2 - \frac{7y^2}{2}.$$

Solution. In order to use the second derivative test, we need to calculate f_{xx}, f_{yy}, and f_{xy}. And before that, we need f_x and f_y to both be 0. These calculations should be pretty routine:

$$\begin{aligned} f_x &= 8 - 6x \\ f_y &= -5 - \frac{14}{2}y = -5 - 7y \\ f_{xx} &= -6 \\ f_{yy} &= -7 \\ f_{xy} &= 0 \end{aligned}$$

Setting $f_x = 8 - 6x = 0$, we get $x = 4/3$, and setting $f_y = -5 - 7y = 0$, we get $y = 5/7$. Thus we only have one critical point, $(4/3, -5/7)$. Plugging the second partials into the SDT,

$$D = (-6)(-7) - 0^2 = 42 > 0.$$

at the critical points $(4/3, -5/7)$, we have $D > 0$ and $f_{xx} < 0$. So $(4/3, -5/7)$ is a local maximum. □

> **Remark.** Notice that D didn't depend on any point (a,b) (nor did f_{xx}). When this happens that doesn't mean that every point is a local maximum—that would mean our function is constant! *The second derivative test applies only to critical points.*

The previous example was a convenient one, but sometimes finding the critical points requires a little more algebraic finesse.

Example 2. Let h be a function whose first order partial derivatives are

$$h_x = -2x + y \quad \text{and} \quad h_y = x - \frac{9}{32}y^3.$$

Find and classify the critical points of h.

Solution. We're already given the first-order partials, setting each of them equal to 0:

$$\begin{aligned} 0 &= -2x + y \\ 2x &= y \end{aligned} \tag{1}$$

and

$$\begin{aligned} 0 &= x - \frac{9}{32}y^3 \\ x &= \frac{9}{32}y^3 \end{aligned} \tag{2}$$

Since we know from (1) that $y = 2x$, we can substitute this into (2) and then solve for x this way:

$$\begin{aligned} x &= \frac{9}{32}(2x)^3 \\ x &= \frac{9}{32} \cdot 8x^3 \\ 0 &= \frac{9}{4}x^3 - x \\ 0 &= x\left(\frac{9}{4}x^2 - 1\right) \end{aligned}$$

This gives one solution of $x = 0$, and for the other we have

$$\begin{aligned} 0 &= \frac{9}{4}x^2 - 1 \\ 1 &= \frac{9}{4}x^2 \\ \frac{4}{9} &= x^2 \\ \pm\frac{2}{3} &= x. \end{aligned}$$

So we have critical points at $x = 0$, $x = 2/3$ and $x = -2/3$. We still need to find the y-values at these points, and for this we use (1). Plugging these x-values in, we find our three critical points: $(0,0)$, $(2/3, 4/3)$ and $(-2/3, -4/3)$.

Next we need to calculate the second partial derivatives and use the SDT.

$$\begin{aligned} f_{xx} &= -2 \\ f_{yy} &= -\frac{27}{32}y^2 \\ f_{xy} &= 1, \end{aligned}$$

so

$$\begin{aligned} D &= (-2)\left(-\frac{27}{32}y^2\right) - 1^2 \\ &= \frac{27}{16}y^2 - 1. \end{aligned}$$

At each point:

$$\begin{aligned} &(0,0)\colon \quad D = -1 < 0 \\ &(2/3, 4/3)\colon \quad D = \frac{27}{16}\left(\frac{4}{3}\right)^2 - 1 = 3 - 1 = 2 > 0 \\ &(-2/3, -4/3)\colon \quad D = \frac{27}{16}\left(-\frac{4}{3}\right)^2 - 1 = 3 - 1 = 2 > 0 \end{aligned}$$

So at $(0,0)$ we have a saddle point. And since $f_{xx} = -2 < 0$, we see that $(2/3, 4/3)$ and $(-2/3, -4/3)$ are both local maxima. □

Example 3. Find and classify the critical points of the function

$$g(u, v) = 3u^2v + 48uv + 4v^2.$$

Solution. Calculating first partial derivatives:

$$\begin{aligned} g_u &= 6uv + 48v \\ g_v &= 3u^2 + 48u + 8v. \end{aligned}$$

Setting $g_u = 0$,

$$\begin{aligned} 0 &= 6uv + 48v \\ 0 &= 6v(u + 8), \end{aligned}$$

which tells us that either $v = 0$ or $u = -8$. If $v = 0$, setting $g_v = 0$ gives

$$\begin{aligned} 0 &= 3u^2 + 48u \\ 0 &= 3u(u + 16), \end{aligned}$$

which tells us $u = 0$ or $u = -16$. This gives us two critical points of $(0,0)$ and $(-16, 0)$. If $u = -8$, setting $g_v = 0$ gives

$$\begin{aligned} 0 &= 3(-8)^2 + 48(-8) + 8v \\ 0 &= 192 - 384 + 8v \\ 8v &= 192 \\ v &= 24, \end{aligned}$$

so our final critical point is $(-8, 24)$. Now finding D in the SDT,

$$\begin{aligned} g_{uu} &= 6v \\ g_{vv} &= 8 \\ g_{uv} &= 6u + 48. \end{aligned}$$

So,

$$\begin{aligned} D &= (6v)(8) - (6u + 48)^2 \\ D(0,0) &= 0 - 48^2 < 0 \\ D(-16,0) &= 0 - (6(-16) + 48)^2 < 0 \\ D(-8,24) &= (6 \cdot 24)(8) - \big(6(-8) + 48\big)^2 > 0 \end{aligned}$$

So $(0, 0)$ and $(-16, 0)$ are saddle points. Since $D(-8, 24) > 0$, we look at $g_{uu}(-8, 24) = 6 \cdot 24 > 0$, which tells us that $(-8, 24)$ is a relative minimum. □

Example 4. Find and classify the critical points of the function

$$f(x, y) = 54x^4 + 64x + \frac{16}{3}y^3 - y + 2.$$

Solution. Again we start by calculating the first partial derivatives and setting them equal to zero.

$$\begin{aligned} f_x &= 216x^3 + 64 \\ f_y &= 16y^2 - 1. \end{aligned}$$

Setting $216x^3 + 64 = 0$ gives $x = -2/3$, and setting $16y^2 - 1 = 0$ gives $y = \pm 1/4$. Putting these together gives us two critical points: $(-2/3, 1/4)$ and $(-2/3, -1/4)$. Next the second partial derivatives:

$$\begin{aligned} f_{xx} &= 648x^2 \\ f_{yy} &= 32y \\ f_{xy} &= 0. \end{aligned}$$

So then

$$D = 648x^2 \cdot 32y - 0.$$

Plugging in our critical points, we find $D(-2/3, 1/4) > 0$. Since $f_{xx}(-2/3, 4/3) > 0$, this tells us that $(-2/3, 4/3)$ is a local minimum. And $D(-2/3, -4/3) < 0$ tells us that $(-2/3, -4/3)$ is a saddle point. □