Overview

In the past two lessons we were concerned with finding maxima and minima of multivariate functions. A common type of problem is maximizing/minimizing a function given some constraint, for example minimizing surface area given a particular volume. For these types of questions there is a more methodical approach with the method of Lagrange multipliers.

Lesson

The basic setup for using Lagrange multipliers is that we are given a function $f(x, y)$ subject to some constraint $g(x, y) = k$, and we want to maximize or minimize f with the given constraint. We'll first describe the method in full detail then see it in action with a few examples.

Method of Lagrange Multipliers. Suppose that $g_x(x, y)$ and $g_y(x, y)$ aren't both zero whenever $g(x, y) = k$. Then to find the maximum and minimum values of $f(x, y)$ subject to the constraint $g(x, y) = k$ (assuming that they exist), do the following.

1. Find all values x, y, and λ (λ is a real number) such that

$$\begin{aligned} f_x &= \lambda g_x \\ f_y &= \lambda g_y \\ g(x, y) &= k \end{aligned}$$

2. Evaluate f at every point (x, y) found in Step 1. The largest of these is the maximum value of f and the smallest is the minimum value.

> **Remark.** Notice that this method is assuming that the maxima and minima exist. They may not exist. This is a subtle point which is handled for you in the statement of the problems you will come across.

Example 1. Find the maximum value of the function

$$f(x, y) = e^{8xy}$$

subject to the constraint $x^2 + y^2 = 100$. Assume both x and y are positive.

Solution. Here $g(x, y) = x^2 + y^2 = 100$. Using Lagrange multipliers, we have

$$\begin{aligned} f_x &= 8ye^{8xy} = \lambda 2x = \lambda g_x && (1) \\ f_y &= 8xe^{8xy} = \lambda 2y = \lambda g_y && (2) \\ g(x, y) &= x^2 + y^2 = 100 && (3) \end{aligned}$$

Solving (1) for λ, we get

$$\begin{aligned} \lambda 2x &= 8ye^{8xy} \\ \lambda &= \frac{4}{x} ye^{8xy}. \end{aligned}$$

Note that we are allowed to divide by x because $x > 0$ by assumption (in particular, $x \neq 0$). Plugging this into λ in (2),

$$\begin{aligned} 8xe^{8xy} &= \lambda 2y \\ 8xe^{8xy} &= \frac{4}{x}ye^{8xy}2y \\ 8x &= \frac{8}{x}y^2 \\ x^2 &= y^2. \end{aligned}$$

Now we use (3):

$$\begin{aligned} x^2 + y^2 &= 100 \\ x^2 + x^2 &= 100 \\ 2x^2 &= 100 \\ x^2 &= 50. \end{aligned}$$

Since x and y are both positive (by assumption), $x^2 = y^2$ implies that $x = y$. Since we found that $x^2 = 50$, this means that $xy = 50$. Thus the maximum value of f is $e^{8\cdot 50} = e^{400}$. □

Example 2. Find the points at which the minimum values of $f(x, y) = x^2e^{y^2}$ subject to the constraint $4y^2 + 2x = 10$ occur.

Solution. We start by setting $f_x = \lambda g_x$ and $f_y = \lambda g_y$.

$$f_x = 2xe^{y^2} = 2\lambda \tag{4}$$

$$f_y = 2yx^2e^{y^2} = 8y\lambda \tag{5}$$

Solving for λ in (4), we get $\lambda = xe^{y^2}$. Using this in (5),

$$\begin{aligned} 2yx^2\cancel{e^{y^2}} &= 8yx\cancel{e^{y^2}} && (e^{y^2} \text{ is never } 0) \\ 2yx^2 &= 8yx \\ 0 &= 8yx - 2yx^2 \\ 0 &= 2yx(4 - x), \end{aligned}$$

which gives us solutions of $x = 0, x = 4$, or $y = 0$. Since we are looking for where the minimum of f occurs, we can immediately see that it must be at $x = 0$. Since e^{y^2} is always positive, the smallest f can possibly be is 0, and $f(0, y) = 0$ for any y. Plugging this into $g(x, y) = 10$, for $x = 0$,

$$\begin{aligned} 4y^2 + 2 \cdot 0 &= 10 \\ y^2 &= \frac{5}{2} \\ y &= \pm\sqrt{\frac{5}{2}}. \end{aligned}$$

In case you're not convinced that we've already found where the minimum occurs, when $x = 4$, we must have

$$\begin{aligned} 4y^2 + 8 &= 10 \\ 4y^2 &= 2 \\ y^2 &= \frac{1}{2} \\ y &= \pm\frac{1}{\sqrt{2}}, \end{aligned}$$

and when $y = 0$, we have

$$\begin{aligned} 2x &= 10 \\ x &= 5. \end{aligned}$$

But $f(4, \pm 1/\sqrt{2}) > f(0, \pm\sqrt{5/2})$ and $f(5, 0) > f(0, \pm\sqrt{5/2})$. Thus the minimum value occurs at $(0, \sqrt{5/2})$ and $(0, -\sqrt{5/2})$. This is already an example where the maximum value does not exist. You'll notice that $f(4, \pm 1/\sqrt{2})$ gives the largest value of f out of these points, but we can easily find other points (x, y) such that $4y^2 + 2x = 10$ but $f(x, y) > f(4, \pm 1/\sqrt{2})$. □

Remark. It is important to make sure that you're never dividing by 0 when eliminating expressions with variables in them. The best way to avoid this is to move everything to one side and factor out as much as you can.

Example 3. Find the maximum of $f(x, y) = \ln(9xy^2)$ subject to the constraint $3x^2 + 8y^2 = 4$.

Solution. First we compute f_x and f_y and set them equal to λg_x and λg_y, respectively. Here $g(x, y) = 3x^2 + 8y^2$.

$$f_x = \frac{9y^2}{9xy^2} = \frac{1}{x} = 6\lambda x = \lambda g_x \tag{6}$$

$$f_y = \frac{18xy}{9xy^2} = \frac{2}{y} = 16\lambda y = \lambda g_y. \tag{7}$$

We can solve (6) for λ by dividing both sides by x. Note that this is allowed since $x \neq 0$ as $x = 0$ would give $f(0, y) = \ln 0$. So $\lambda = \frac{1}{6x^2}$. Plugging this into (7),

$$\begin{aligned} \frac{2}{y} &= 16\lambda y \\ \frac{2}{y} &= 16\left(\frac{1}{6x^2}\right) y \\ 12x^2 &= 16y^2 \\ x^2 &= \frac{16}{12}y^2 \\ x^2 &= \frac{4}{3}y^2. \end{aligned} \tag{8}$$

We can plug this into our constraint equation to get

$$\begin{aligned} 3x^2 + 8y^2 &= 4 \\ 3\left(\frac{4}{3}y^2\right) + 8y^2 &= 4 \\ 4y^2 + 8y^2 &= 4 \\ 12y^2 &= 4 \\ y^2 &= \frac{1}{3} \\ y &= \pm\frac{1}{\sqrt{3}}. \end{aligned}$$

Now using this in (8),

$$\begin{aligned} x^2 &= \frac{4}{3} \cdot \frac{1}{3} \\ x^2 &= \frac{4}{9} \\ x &= \pm\frac{2}{3}. \end{aligned}$$

But $x = -2/3$ is not possible since that would force us to take ln of a negative number. Thus our solutions are $(2/3, 1/\sqrt{3})$ and $(2/3, -1/\sqrt{3})$. Notice that $f(2/3, 1/\sqrt{3}) = f(2/3, -1/\sqrt{3})$ since we're squaring y. So we have a maximum of $\ln\left(9 \cdot \frac{2}{3} \cdot \frac{1}{3}\right) = \ln 2$. □

Example 4. Find the minimum value of $f(x, y) = x^2 + y^2$ subject to the constraint $5y = 5 - 2x$.

Solution. Recall that in the method of Lagrange multipliers we need our constraint to be of the form $g(x, y) = k$. But this is no problem here since adding $2x$ to both sides of our constraint equation gives us $g(x, y) = 2x + 5y = 5$.

Setting $f_x = \lambda g_x$ and $f_y = \lambda g_y$,

$$f_x = 2x = 2\lambda = \lambda g_x \tag{9}$$

$$f_y = 2y = 5\lambda = \lambda g_y. \tag{10}$$

Right away in (9), we can see that $\lambda = x$. Plugging this into (10), we get

$$\begin{aligned} 2y &= 5\lambda \\ 2y &= 5x \\ y &= \frac{5}{2}x. \end{aligned}$$

Plugging this into our constraint,

$$\begin{aligned}
2x + 5y &= 5 \\
2x + 5\left(\frac{5}{2}x\right) &= 5 \\
\frac{4}{2}x + \frac{25}{2}x &= 5 \\
\frac{29}{2}x &= 5 \\
x &= \frac{10}{29}.
\end{aligned}$$

Plugging this into our equation for y, we find $y = \frac{25}{29}$. Thus our minimum is

$$\left(\frac{10}{29}\right)^2 + \left(\frac{25}{29}\right)^2 = \frac{25}{29}.$$

□