Overview

The last calculus topic we'll discuss is double integrals. In this lesson we briefly cover how to compute them and delve into more detail in the following lessons.

Lesson

The idea for computing double integrals is very simple. Just as with partial derivatives we treat one variable at a time. There are two possible orders of integration we can encounter: $dx\,dy$ or $dy\,dx$. In either case we consider the inside integral first.

For the inside integral below we treat y as a constant and integrate with respect to x first in order to obtain a function of y only.

$$\int_c^d \int_a^b f(x,y)\,dx\,dy = \int_c^d \left(\int_a^b f(x,y)\,dx \right) dy$$

And for the inside integral below here, we treat x as a constant while integrating with respect to y first and then obtain a function of x only.

$$\int_a^b \int_c^d f(x,y)\,dy\,dx = \int_a^b \left(\int_c^d f(x,y)\,dy \right) dx$$

The best way to get the hang of this is with examples.

Example 1. Evaluate the double integral.

$$\int_0^9 \int_0^{\sqrt{2}} 3xy\,dx\,dy$$

Solution.

$$\begin{aligned}
\int_0^9 \left(\int_0^{\sqrt{2}} 3xy\,dx \right) dy &= \int_0^9 \left(\frac{3}{2}x^2 y \Big|_{x=0}^{x=\sqrt{2}} \right) dy \\
&= \int_0^9 3y\,dy \\
&= \frac{3}{2}y^2 \Big|_0^9 \\
&= \frac{243}{2}
\end{aligned}$$

□

Example 2. Compute.

$$\int_3^4 \int_2^4 3x^3y^2\,dy\,dx$$

Solution.

$$\begin{aligned}\int_3^4 \int_2^4 3x^3y^2 \, dy \, dx &= \int_3^4 \left(x^3y^3 \Big|_{y=2}^{y=4} \right) dx \\ &= \int_3^4 x^3(4^3 - 2^3) \, dx \\ &= \int_3^4 56x^3 \, dx \\ &= 14x^4 \Big|_3^4 \\ &= 14(4^4 - 3^4) \\ &= 2450\end{aligned}$$

□

Recall that in one-variable calculus computing a definite integral results in a number. But as we've seen, computing the inside integral produces a function. So there's really nothing special about the limits of integration of the inside integral being numbers. For the inside integral of $\int_c^d \int_a^b f(x,y) \, dx \, dy$, y is a constant with respect to x. So instead of a and b just being numbers, they could actually be functions of y.

Example 3. Evaluate the double integral.

$$\int_5^6 \int_0^y 8xy \, dx \, dy$$

Solution.

$$\begin{aligned}\int_5^6 \left(\int_0^y 8xy \, dx \right) dy &= \int_5^6 \left(4x^2y \Big|_0^y \right) dy \\ &= \int_5^6 4y^3 \\ &= y^4 \Big|_5^6 \\ &= 6^4 - 5^4 \\ &= 671.\end{aligned}$$

□

The same goes if the roles of x and y are switched in the discussion above the previous example.

Example 4. Compute.

$$\int_0^{\sqrt{\pi/2}} \int_0^{x^2} -4x \cos y \, dy \, dx$$

Solution.

$$\int_0^{\sqrt{\pi/2}} \left(\int_0^{x^2} -4x \cos y \, dy \right) dx = \int_0^{\sqrt{\pi/2}} -4x \sin x^2 \, dx \qquad \begin{aligned} u &= x^2 \\ du &= 2x\,dx \end{aligned}$$
$$= 2 \int_0^{\pi/2} -\sin u \, du$$
$$= 2 \left(\cos \frac{\pi}{2} - \cos 0 \right)$$
$$= -2. \qquad \square$$

Example 5. Compute the integral.

$$\int_1^e \int_0^{9 \ln x} 5x \, dy \, dx$$

Solution.

$$\int_1^e \left(\int_0^{9 \ln x} 5x \, dy \right) dx = \int_1^e \left(5xy \Big|_0^{9 \ln x} \right) dx$$
$$= 45 \int_1^e x \ln x \, dx$$
$$= 45 \int_1^e \underbrace{(\ln x)}_{u} \underbrace{x \, dx}_{dv}$$
$$= 45 \left(\frac{1}{2} x^2 \ln x - \int \frac{1}{2} x \, dx \right) \Big|_1^e$$
$$= 45 \left(\frac{1}{2} x^2 \ln x - \frac{1}{4} x^2 \right) \Big|_1^e$$
$$= 45 \left[\left(\frac{1}{2} e^2 \ln e - \frac{1}{4} e^2 \right) - \left(\frac{1}{2} 1^2 \ln 1 - \frac{1}{4} \cdot 1^2 \right) \right]$$
$$= \frac{45}{4} \left(e^2 + 1 \right). \qquad \square$$