Overview

In the previous lesson we discussed how to calculate a double integral. In this lesson we will look more into what the double integral is and Fubini's Theorem.

Lesson

In one-variable calculus we define the definite integral as a limit of Riemann sum. In two variables we can do a very similar thing. If you want some easy-to-read details on this, consult Paul's Online Math Notes. Here we'll just discuss the notation and what to do with it. Given a nonnegative function $f(x, y)$ and a region D in the xy-plane, then the double integral

$$\iint_D f(x, y)\, dA$$

can be interpreted as the *volume* under the surface $z = f(x, y)$ and above the region D. (In the case of any $f(x, y)$, we can think of the integral as a *signed volume*.) This region D has a name, and it is called the *domain of integration* (hence the letter D).

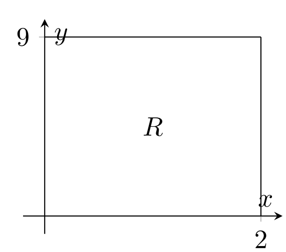
But this conceptual idea of the integral is not something that we can calculate, so we need to turn such an integral into the type that we saw in the previous lesson. Let's start with a rectangle, the simplest domain of integration.

Example 1. Evaluate the integral

$$\iint_R 10x^3 y\, dA,$$

where R is the rectangle with vertices $(0, 0), (2, 0), (0, 9)$ and $(2, 9)$.

Solution. We should always start by drawing the domain of integration. This one is pretty simple though.



In R we notice that x ranges from 0 to 2 and y ranges from 0 to 9. So we can write the integral as

$$\int_0^9 \int_0^2 10x^3 y\, dx\, dy.$$

From this point, we should know how to calculate the integral from the previous lesson.

$$\begin{aligned}\int_0^9 \int_0^2 10x^3 y\,dx\,dy &= \int_0^9 \left(\frac{5}{2}x^4 y \Big|_{x=0}^{x=2} \right) dy \\ &= \int_0^9 \frac{5}{2} \cdot 2^4 y\,dy \\ &= \int_0^9 40y\,dy \\ &= 20y^2 \Big|_0^9 \\ &= 20 \cdot 9^2 \\ &= 1620. \end{aligned}$$

□

In the previous example we made a choice in integrating with respect to x first. It would be natural to ask the question does the order of integration matter? There's a famous theorem in calculus that the order of integration usually does not matter.

Theorem (Fubini's Theorem)**.** If f is a continuous function on the rectangle $R = [a, b] \times [c, d]$, then

$$\iint_R f(x, y)\,dA = \int_a^b \int_c^d f(x, y)\,dy\,dx = \int_c^d \int_a^b f(x, y)\,dx\,dy.$$

There are actually more general situations where Fubini's Theorem applies, but such examples won't show up in this class. Let's take a look at an example of switching the order of integration.

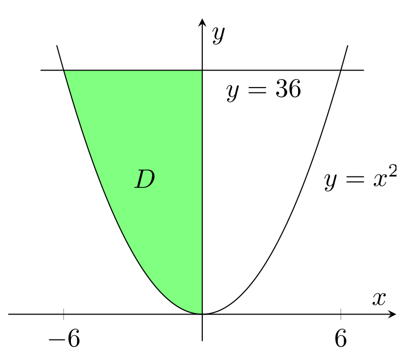
Example 2. Switch the order of integration on the following integral.

$$\int_{-6}^0 \int_{x^2}^{36} f(x, y)\,dy\,dx$$

Solution. Let's call our domain of integration D. Here y ranges from x^2 up to 36 and x ranges from -6 to 0. We could put this in set notation as

$$D = \left\{ (x, y) \colon -6 \le x \le 0,\, x^2 \le y \le 36 \right\}.$$

Putting this same information in a picture:



Looking at the picture, we can see that we can describe the D in another way. The smallest y-value in D is 0 and the largest is 36, so in D $0 \le y \le 36$. As for x, we see that x lies between the parabola and the y-axis. Solving $y = x^2$ for x we get $x = \pm\sqrt{y}$. Since $x \le 0$ in D, we know that we want $x = -\sqrt{y}$. Putting this all together, we can write $D = \left\{(x, y)\colon -\sqrt{y} \le x \le 0,\, 0 \le y \le 36\right\}$. And rewriting the integral in question,

$$\int_0^{36} \int_{-\sqrt{y}}^{0} f(x, y)\, dx\, dy. \qquad \square$$

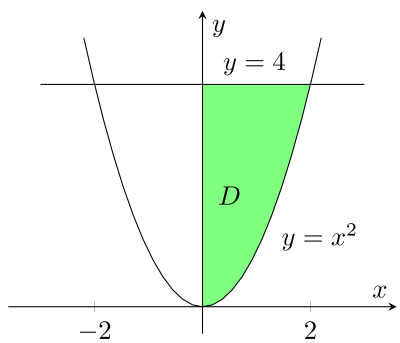
Remark. There is an important distinction between the domain of integration and the function we are integrating. We didn't need to know what the function $f(x, y)$ was explicitly in order to switch the order of integration.

At this point it may seem like switching the order of integration is just a neat little parlor trick. But sometimes it's actually necessary to switch the order as in the following example.

Example 3. Evaluate the following integral.

$$\int_0^2 \int_{x^2}^{4} -3x\sqrt{1 + y^2}\, dy\, dx$$

Solution. If we tried to compute the integral as is we would get stuck since we don't know how to find an antiderivative of $\sqrt{1 + y^2}$. After the previous example, drawing the domain D should be straightforward. In this example $0 \le x \le 2$ and $x^2 \le y \le 4$.



So another way we can describe D is

$$D = \left\{(x, y)\colon 0 \le x \le \sqrt{y},\, 0 \le y \le 4\right\}.$$

Putting this into the double integral and calculating,

$$\begin{aligned}
\int_0^4 \int_0^{\sqrt{y}} -3x\sqrt{1+y^2}\,dx\,dy &= \int_0^4 \left(-\frac{3x^2}{2}\sqrt{1+y^2}\Bigg|_{x=0}^{x=\sqrt{y}} \right) dy \\
&= \int_0^4 -\frac{3y}{2}\sqrt{1+y^2}\,dy && u = 1+y^2 \\
& && du = 2y\,dy \\
&= -\frac{3}{4}\int_1^{17} u^{1/2}\,du \\
&= -\frac{3}{4}\cdot\frac{2}{3}u^{3/2}\Bigg|_1^{17} \\
&= -\frac{1}{2}\left(17^{3/2}-1\right)
\end{aligned}$$

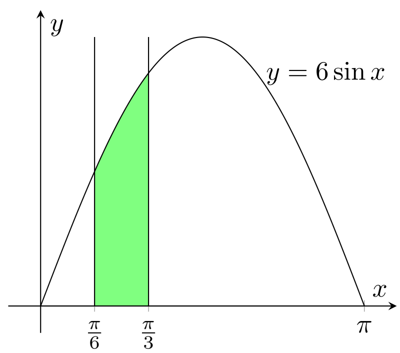
□

Example 4. Evaluate the integral

$$\iint_D 8\sec^2 x\,dy\,dx,$$

where D is the region bounded by $y=0$, $y=6\sin x$, $x=\frac{\pi}{6}$ and $x=\frac{\pi}{3}$.

Solution. As usual, we draw the region D.



The left and right bounds for D are $\frac{\pi}{6}$ and $\frac{\pi}{3}$, respectively, and D is bounded on the bottom by $y=0$ and on the top by $y=6\sin x$. Since the bounds for y include a function of x, we'll want to do dy first. Putting this into the integral,

$$\begin{aligned}
\iint_D 8\sec^2 x\,dA &= \int_{\pi/6}^{\pi/3}\int_0^{6\sin x} 8\sec^2 x\,dy\,dx \\
&= \int_{\pi/6}^{\pi/3}\left(8y\sec^2 x\Big|_0^{6\sin x}\right)dx \\
&= \int_{\pi/6}^{\pi/3} 48\sin x\sec^2 x\,dx \\
&= 48\int_{\pi/6}^{\pi/3}\frac{\sin x}{\cos x}\sec x\,dx \\
&= 48\int_{\pi/6}^{\pi/3}\tan x\sec x\,dx \\
&= 48\sec x\Big|_{\pi/6}^{\pi/3} \\
&= 48\left(\sec\frac{\pi}{3}-\sec\frac{\pi}{6}\right) \\
&= 48\left(2-\frac{2}{\sqrt{3}}\right)
\end{aligned}$$

□

Example 5. This entire lesson has hinged upon being able to switch the order of integration, but using a double integral calculator, we could find

$$\begin{aligned}
\int_0^1\int_0^1 \frac{x-y}{(x+y)^3}\,dy\,dx &= \frac{1}{2} \\
\int_0^1\int_0^1 \frac{x-y}{(x+y)^3}\,dx\,dy &= -\frac{1}{2}.
\end{aligned}$$

So what's going on here? In this example Fubini's Theorem doesn't apply since the function is not continuous at $(0,0)$. Moreover, this shows that we can't *always* switch the order of integration and get the same answer. Although in this class it's safe to assume that we can.