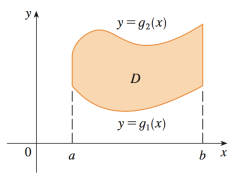
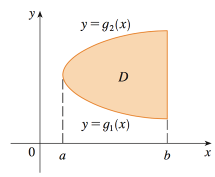
Overview

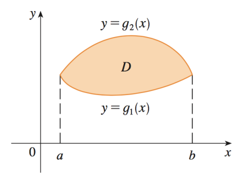
In this lesson we build on the previous two by complicating our domains of integration and discussing the average value of functions of two variables.

Lesson

So far the domains of integration which we have considered have been pretty simple: rectangles and regions bounded by a function and horizontal and vertical lines. We can complicate things slightly by considering two types of regions. The first type is where D is bounded above and below by functions of x. Some examples taken from Stewart's Calculus are shown below.





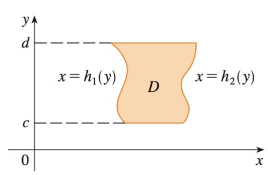


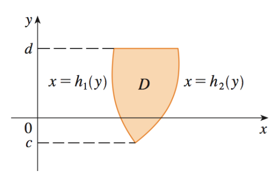
Notice if you follow along from left to right, there is always a top function and a bottom function on the boundary.

Furthermore, since y ranges from $g_1(x)$ to $g_1(y)$, if we encounter this type of integral, we must set it up as

$$\iint_D f(x,y)\,dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx.$$

The second type of region that we encounter is where D is bounded to the left and right by functions of y. A couple examples also from Stewart are given below. You'll notice that following along from bottom to top there is always a left function and a right function on the boundary.





This time you'll notice that x ranges from $h_1(y)$ to $h_2(y)$, so if we encounter a region of this type, we must set up our integral as

$$\iint_D f(x,y)\,dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy.$$

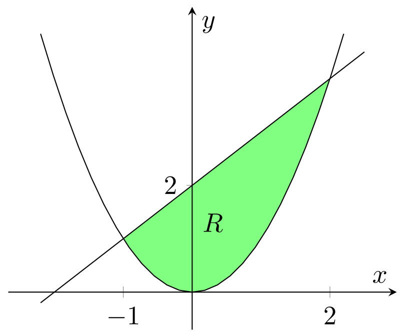
So from now on when given an integral $\iint_D f(x,y)\,dA$, we will have to determine which type of region we have. Let's try it in an example.

Example 1. Evaluate

$$\iint_R (8xy + 7)\, dA$$

where R is the region bounded by $y = x^2$ and $y = x + 2$.

Solution. We start by drawing a picture of our domain of integration R.



Here R is bounded above and below by functions of x, so we will want our inside integral to be "dy." To determine the bounds for x, we want to find the points of intersection of the line and the parabola, and we do this by setting their equations equal to one another.

$$\begin{aligned} x^2 &= x + 2 \\ 0 &= x^2 - x - 2 \\ 0 &= (x+1)(x-2) \end{aligned}$$

Now that we've verified that the two curves intersect at $x = -1$ and $x = 2$, we need to know the range of y in R. The smallest y can be is on the parabola, and the largest y can be is on the line. So $x^2 \le y \le x$. Now we have enough information to compute the double integral.

$$\begin{aligned} &\int_{-1}^{2} \int_{x^2}^{x+2} \big(8xy + 7\big)\, dy\, dx \\ &\qquad = \int_{-1}^{2} \left[\big(4xy^2 + 7y\big)\bigg|_{y=x^2}^{y=x+2} \right] dx \\ &\qquad = \int_{-1}^{2} \big[4x(x+2)^2 + 7(x+2) - (4x^5 + 7x^2)\big]\, dx \\ &\qquad = \int_{-1}^{2} \big(-4x^5 + 4x^3 + 9x^2 + 23x + 14\big)\, dx \\ &\qquad = -\frac{2}{3}x^6 + x^4 + 3x^3 + \frac{23}{2}x^2 + 14x\bigg|_{-1}^{2} \\ &\qquad = 76.5 \end{aligned}$$

□

Sometimes our domain of integration isn't so nice. But there's a nice calculus fact that can help us out.

Big fact. If D can be divided into two parts D_1 and D_2 so that D_1 and D_2 don't overlap except for their boundaries, then

$$\iint_D f(x,y)\,dA = \iint_{D_1} f(x,y)\,dA + \iint_{D_2} f(x,y)\,dA.$$

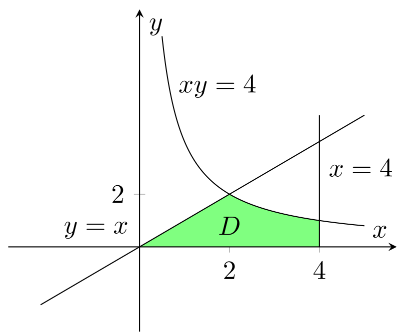
By this fact, if we are given a domain of integration that isn't one of the two nice types we discussed earlier, we can divide it up to make it one of the two types and compute each integral separately.

Example 2. Evaluate

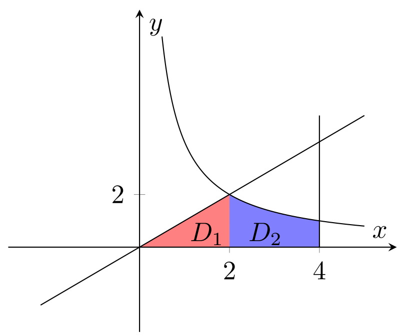
$$\iint_D 4x^2\,dA$$

where D is the region in the first quadrant bounded by the hyperbola $xy = 4$ and the lines $y = x$, $y = 0$ and $x = 4$.

Solution. We start by drawing a picture of D. We should figure out where the curves intersect to help us complete the picture. Using $y = x$ in the equation $xy = 4$, we get $x^2 = 4$, or $x = 2$ (since we are in the first quadrant).



Notice that there is no way to pick out a function that is always on top or always on the right, so we'll have to divide up D. One way to do this is to make a division at $x = 2$.



Now we compute $\iint_{D_1} 4x^2\,dA$ and $\iint_{D_2} 4x^2\,dA$ separately. For the first one,

$$\begin{aligned}\iint_{D_1} 4x^2\,dA &= \int_0^2 \int_0^x 4x^2\,dy\,dx \\ &= \int_0^2 4x^3\,dx \\ &= x^4\Big|_0^2 \\ &= 16.\end{aligned}$$

And for the second,

$$\begin{aligned}\iint_{D_2} 4x^2\,dA &= \int_2^4 \int_0^{4/x} 4x^2\,dy\,dx \\ &= \int_2^4 16x \\ &= 8x^2\Big|_2^4 \\ &= 96.\end{aligned}$$

Now using the big fact, $\iint_D 4x^2\,dA = 16 + 96 = 112$. □

In Lesson 2 we discussed the average value of a function of a single variable. We started with the idea of taking a discrete average: add up all the values and divide by the number of values you have. We extended this by saying that if we have a function f on an interval $[a, b]$, then the average value of f on $[a, b]$ is given by

$$f_{\text{avg}} = \frac{1}{b-a}\int_a^b f(x)\,dx. \tag{1}$$

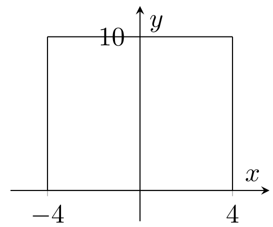
We can extend this notion again to find the average value of a function of two variables. Given a function $f(x, y)$ and a region R we can define the average value of $f(x, y)$ over R as

$$f_{\text{avg}} = \frac{1}{\mathrm{A}(R)}\iint_R f(x,y)\,dA, \tag{2}$$

where $\mathrm{A}(R)$ denotes the area of R. If you think about it, $b - a$ is like the one-dimensional area of $[a, b]$, so this formula isn't really all that different from (1). Of course the simplest such region to deal with is a rectangle as in the next example.

Example 3. Find the average value of the function $f(x, y) = 6x + 4y$ over the rectangle $R = [-4, 4] \times [0, 10]$.

Solution. Let's start by computing $\mathrm{A}(R)$. R is the following rectangle.



So A$(R) = (4-(-4))\cdot(10-0) = 80$. Moreover, x ranges from -4 to 4 and y ranges from 0 to 10. Putting this all together into the formula from (2),

$$\begin{aligned} f_{\text{avg}} &= \frac{1}{80}\int_{-4}^{4}\int_{0}^{10}(6x+4y)\,dy\,dx \\ &= \frac{1}{80}\int_{-4}^{4}\left(6xy+2y^2\bigg|_{y=0}^{y=10}\right)\,dx \\ &= \frac{1}{80}\int_{-4}^{4}(60x+200)\,dx \\ &= \frac{1}{80}\left(30x^2+200x\bigg|_{-4}^{4}\right) \\ &= \frac{1}{80}1600 \\ &= 20. \end{aligned}$$

□

Recall from Calculus I that if we have a nonnegative function f on an interval $[a,b]$, then we can interpret the definite integral $\int_a^b f(x)\,dx$ as the are under the curve on the given interval. We have a similar notion in functions of two variables. That is if $f(x,y)$ is nonnegative over a region R, then the integral

$$\iint_R f(x,y)\,dA$$

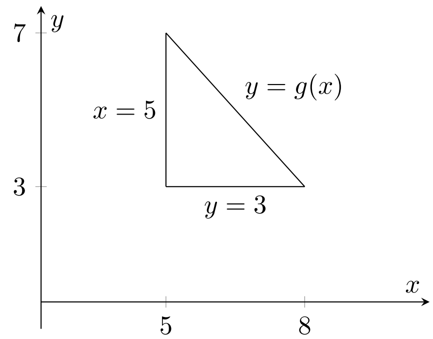
is interpreted as the *volume* under the surface $z = f(x,y)$ and above R.

Example 4. Find the volume under the surface $z = xy$ above the triangle T whose vertices are $(5,3,0)$, $(8,3,0)$ and $(5,7,0)$.

Solution. Since the surface $z = xy$ lives in a 3-dimensional space, we can talk about T as living in 3 dimensions as well. But notice that the z-coordinate of each vertex of T is 0, so T lives in the xy-plane. Note that by the discussion above, what we're after is

$$\iint_T xy\,dA.$$

We should draw a picture.



If we knew an equation for the hypotenuse of the triangle, then we would be good to go. In order to do this recall the point-slope form of a line

$$y = m(x - x_1) + y_1 \tag{3}$$

where m is the slope of the line and (x_1, y_1) is a particular point on the line. To calculate the slope we use

$$m = \frac{\Delta y}{\Delta x} = \frac{3-7}{8-5} = -\frac{4}{3}.$$

Then we pick a point on the line and plug it into (3).

$$\begin{aligned} y &= -\frac{4}{3}(x-5) + 7 \\ &= -\frac{4}{3}x + \frac{20}{3} + \frac{21}{3} \\ &= -\frac{4}{3}x + \frac{41}{3}. \end{aligned}$$

Now, from the picture we see that y ranges from 3 to the line $y = -\frac{4}{3} + \frac{41}{3}$ while x ranges from 5 to 8. Putting this into a double integral,

$$\begin{aligned}
\iint_T xy\,dA &= \int_5^8 \int_3^{-\frac{4}{3}x+\frac{41}{3}} xy\,dy\,dx \\
&= \int_5^8 \left(\frac{1}{2}xy^2 \bigg|_{y=3}^{y=-\frac{4}{3}x+\frac{41}{3}} \right) dx \\
&= \int_5^8 \left(\frac{1}{2}x\left(-\frac{4}{3}x+\frac{41}{3}\right)^2 - \frac{9}{2} \right) dx \\
&= \int_5^8 \left[\frac{1}{2}x\left(\frac{16}{9}x^2 - \frac{328}{9}x + \frac{1681}{9}\right) - \frac{9}{2}x \right] dx \\
&= \int_5^8 \left(\frac{8}{9}x^3 - \frac{164}{9}x^2 + \frac{800}{9}x \right) dx \\
&= \frac{2}{9}x^4 - \frac{164}{27}x^3 + \frac{400}{9}x^2 \bigg|_5^8 \\
&= \frac{17408}{27} - \frac{13250}{27} \\
&= 154.
\end{aligned}$$

□