Overview

In this lesson we devote our attention to the natural logarithm, written $\ln x$. Recall that ln is precisely the inverse function of e^x. That is $e^x = y$ if and only if $\ln y = x$. There are several properties of logarithms that we should recall from elementary algebra or precalculus.

- $\ln 1 = 0$
- $\ln(ab) = \ln a + \ln b$
- $\ln\left(\frac{a}{b}\right) = \ln a - \ln b$
- $\ln(a^b) = b \ln a$
- $y = \ln x$ has domain $(0, \infty)$ and range $(-\infty, \infty)$
- $(\ln x)' = \frac{1}{x}$, which implies that $\ln x$ is always increasing on its domain.
- By the chain rule, for any $u = u(x)$ (u is a function of x), we have that $(\ln u)' = \frac{u'}{u}$.

You should also be very familiar with the exponential e^x. If you are rusty on these properties, look at the Lesson 17 link for MA 158 (Fall 2016), or use one of the many resources easily found via Google. One thing we should stress is that when we see e by itself, that is just a number approximately equal to 2.71828. For example, if we ever see something like

$$\int e^4 \, dx,$$

we should be thinking *power rule*, not exponential. And the result is $e^4 x + C$.

Lesson

The problem at hand is the following. We want to be able to compute

$$\int_1^x \frac{1}{t} \, dt = \int_1^x t^{-1} \, dt.$$

The natural guess is to apply the power rule, but that gets us in trouble as we would obtain

$$\frac{1}{-1+1} t^{-1+1} + C = \frac{1}{0} + C,$$

which is undefined. It turns out that the function that arises as the antiderivative is the natural log. So

$$\boxed{\ln x = \int_1^x \frac{1}{t} \, dt}$$

Generally, we will need to throw on absolute values for the argument of ln because of its domain restriction.

Remark. We only want to add the absolute value when necessary for answers in Loncapa.

Example 1. Compute
$$I = \int_{\sqrt{\pi/2}}^{\sqrt{3\pi}/2} 2x\cot(x^2)\,dx.$$

Solution. The trick here is to write $\cot x^2$ in terms of $\sin x^2$ and $\cos x^2$. Remember that $\cot\alpha = \frac{\cos\alpha}{\sin\alpha}$. Thus
$$\begin{aligned}
I &= \int_{\sqrt{\pi/2}}^{\sqrt{3\pi}/2} 2x \cdot \frac{\cos x^2}{\sin x^2}\,dx. && u = \sin x^2 \\
&= \int_1^{1/\sqrt{2}} \frac{du}{u} && du = 2x\cos x^2\,dx \\
&= \ln u\Big|_1^{1/\sqrt{2}} \\
&= \ln\frac{1}{\sqrt{2}} - \ln 1 \\
&= \ln\frac{1}{\sqrt{2}}
\end{aligned}$$
□

The following example illustrates an important strategy in u-substitution problems.

Example 2. Compute
$$I = \int \frac{9\cos(\ln 9x)}{x}\,dx.$$

Solution. Since we have just learned that $\int \frac{1}{x}\,dx = \ln x + C$, it may be tempting to use that somehow in this problem. But remember that our focus is to pick a u whose derivative works out nicely. In some sense we do end up using that fact as we want to pick
$$\begin{aligned}
u &= \ln 9x \\
du &= \frac{1}{x}\,dx.
\end{aligned}$$

Then
$$\begin{aligned}
I = 9\int \cos\Big(\underbrace{\ln 9x}_{u}\Big) \cdot \underbrace{\frac{1}{x}\,dx}_{du} &= 9 \\
&= 9\int \cos u\,du \\
&= 9\sin u + C \\
&= 9\sin(\ln 9x) + C.
\end{aligned}$$
□

Remark. We don't need to put $\ln|9x|$ since $\ln 9x$ appeared in the original function. That means that the original function had a domain of $(0,\infty)$, so the domain of the antiderivative is unaffected.

Example 3. Compute
$$I = \int \frac{10}{x^{1/3}\left(7 + x^{2/3}\right)}\,dx.$$

Solution. This problem requires a similar strategy to one we've seen before. The only difference is the integral that we obtain in terms of u. For let

$$\begin{aligned} u &= 7 + x^{2/3} \\ du &= \frac{2}{3}x^{-1/3}\,dx \\ &\boxed{\frac{3}{2}du = x^{-1/3}\,dx.} \end{aligned}$$

Then

$$\begin{aligned} I &= 10 \cdot \frac{3}{2}\int \frac{1}{u}\,du \\ &= 15\int \frac{du}{u} \\ &= 15\ln|u| + C \\ &= 15\ln\left|7 + x^{2/3}\right| + C \\ &= 15\ln\left(7 + x^{2/3}\right) + C. \end{aligned}$$

□

Example 4. According to demographers, the population of Accident, Maryland grew at a rate of

$$P'(t) = \frac{30e^t}{10+e^t} \quad \text{hundreds of people per year}$$

from the time the town was established until its ironic accidental demise 7 years later. If there were 6,000 people at the town's inception, how many people lived in Accident at the time of the accident?

Solution. The population $P(t)$ is given by

$$\begin{aligned} P(t) &= \int \frac{30e^t}{10+e^t}\,dt \\ &= 30\int \frac{du}{u} && u = 10 + e^t \\ &= 30\ln|u| + C && du = e^t\,dt \\ &= 30\ln(10+e^t) + C. && (10 + e^t > 0 \text{ for all } t) \end{aligned}$$

Now using that $P(0) = 6000$,

$$\begin{aligned} 6000 &= 30\ln(10+e^0) + C \\ &\boxed{C = 6000 - 30\ln(11).} \end{aligned}$$

So

$$P(t) = 30\ln(10+e^t) + 6000 - 30\ln(11).$$

Now the question is simply asking what is $P(7)$. Well,

$$\begin{aligned} P(7) &= 30\ln(10+e^7) + 6000 - 30\ln(11) \\ &\approx 6138. \end{aligned}$$

□

Example 5. Find the area of the region bounded by the curves

$$y = \frac{3}{2x \ln \sqrt{x}}, \quad x = e, \quad x = e^3, \quad \text{and } y = 0.$$

Solution. We set up the integral as follows:

$$I = \int_e^{e^3} \frac{3}{2x \ln \sqrt{x}}\, dx.$$

Unsurprisingly, this requires a u-substitution. Let $u = \ln \sqrt{x}$. Then $du = \frac{1}{2x}\, dx$. Now

$$\begin{aligned} I &= 3 \int_{x=e}^{x=e^3} \frac{du}{u} \\ &= 3 \int_{1/2}^{3/2} \frac{du}{u} \\ &= 3(\ln u)\Big|_{1/2}^{3/2} \\ &= 3\left(\ln \frac{3}{2} - \ln \frac{1}{2}\right) \\ &= 3(\ln 3 - \ln 2 - (\ln 1 - \ln 2)) \\ &= 3(\ln 3 - \ln 2 + \ln 2) \\ &= 3 \ln 3. \end{aligned}$$

□