Overview

In this lesson we make the natural transition from multivariate calculus to some basics in linear algebra. We start with using matrices to solve systems of linear equations and some easy application problems.

Lesson

Systems of linear equations

Much of this lesson may well be review from middle school or high school math classes, but for the sake of completeness we'll go through everything in full detail. As we begin a completely new topic, it is necessary to go through some definitions.

Just as we saw in the section on differential equations, an equation is said to be *linear* if no two variables are multiplied together. For example, $5x + 2y - z = 3$ is linear, but $3xy - z = 1$ is not. A *system of linear equations* is a group of two or more equations with the same set of unkowns (variables). Generally they are grouped together with a left curly brace. For example, the following is a system of two equations in three unkowns.

$$\begin{cases} 3x + 7y + 28z = 4 \\ 19x + 5y + 2z = 13 \end{cases}$$

Note. We will generally abbreviate "system of linear equations" to "system of equations" or simply "system," as it should be understood that the systems we are dealing with are all linear.

Solving systems of linear equations

Most of the problems we'll come across will be systems in two or three unkowns, so we'll keep our examples to those types. In the case of systems in two unkowns, the best practice is to use substitution or elimination as you learned in middle school. The process of using matrices that we will soon describe is overkill for such a scenario.

A matrix is simply an array of numbers, and we can use such an object to represent a system of linear equations. To do this, we use an *augmented matrix*, which consists of a matrix followed by a vertical line and then a single column of numbers. Given a system

$$ax + by + cz = d$$
$$ex + fy + gz = h$$
$$ix + jy + kz = l$$

we write this in a matrix as

$$\left[\begin{array}{ccc|c} a & b & c & d \\ e & f & g & h \\ i & j & k & l \end{array} \right]$$

So the each column corresponds to the coefficients of x, y and z, respectively, followed by the column of what each equation is equal to. If any of the coefficients are negative, we include that too.

Example 1. Write the augmented matrix that represents the following system of equations.
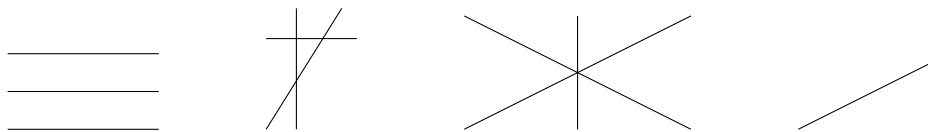
$$-2x - y = 1$$
$$2x - 4y - 5z = -2$$
$$-x + 2y + 4z = -4$$

Solution.

$$\left[\begin{array}{ccc|c} -2 & -1 & 0 & 1 \\ 2 & -4 & -5 & -2 \\ -1 & 2 & 4 & -4 \end{array} \right]$$

□

Solutions to systems of equations

A system is said to be *consistent* if a solution exists, and *inconsistent* if there is no solution. Moreover, a consistent system is said to be *independent* if there is *exactly one* solution, and *dependent* if there is *more than one* solution. Since we are dealing with linear things here, it should be easy to convince yourself that in the case of more than one solution we actually have infinitely many solutions. Thus the only options for number of solutions to linear systems are zero, one or infinitely many solutions.

Graphically speaking, a solution is where the lines or planes intersect. Below, we have examples of systems of three equations in two unkowns. The first two are inconsistent. The third is a consistent independent system and the last is a consistent dependent system. (Imagine there are three lines on top of each other.)

Algebraically speaking, an inconsistent system comes down to a statement of the form $0 = a \neq 0$. For example the augmented matrix,

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 4 \\ 0 & 1 & 3 & -2 \\ 0 & 0 & 0 & 1 \end{array} \right]$$

represents an inconsistent system of equations, because the last row says that $0x + 0y + 0z = 1$, that is, $0 = 1$. A system will be independent if the corresponding matrix has the form

$$\left[\begin{array}{ccc|c} 1 & * & * & a \\ 0 & 1 & * & b \\ 0 & 0 & 1 & c \end{array} \right]$$

and the system will be dependent if the corresponding augmented matrix has the form

$$\left[\begin{array}{ccc|c} 1 & * & * & a \\ 0 & 1 & * & b \\ 0 & 0 & 0 & 0 \end{array} \right].$$

If this is the case, then we say that z is a *free variable*. The free variable is corresponds to the column without the leading 1. When a situation like this arises, we set $z = t$ (or whatever the free variable is), and then solve for x and y in terms of $z = t$.

How do we get matrices to look like those above? To use matrices in order to solve systems of equations, we use a process called *Gaussian elimination*. In this process, our goal is to obtain a matrix of the form

$$\left[\begin{array}{ccc|c} 1 & * & * & a \\ 0 & 1 & * & b \\ 0 & 0 & 1 & c \end{array}\right]$$

We don't care what happens in the spots marked with a *, but what we want is a 1 in the first entry of each row, and all 0's below every leading 1. Such a matrix is said to be in *row echelon form*. Note that we may not always get all 1's on the diagonal if we end up with a row of 0's. In that case, refer to the matrix forms mentioned above. The algorithm we follow goes like this.

Gaussian elimination

1. Get a 1 in the top left entry.

2. Make every entry below this 1 a 0.

3. Go to the next row and find the first nonzero entry. Make this a 1.

4. Make every entry below this 1 a 0.

5. Repeat this process until you run out of rows.

6. Translate the matrix back into equations to solve.

Now we need to know what are the valid operations to make this happen. These are referred to as the elementary row operations.

Elementary row operations

1. Switch any two rows.

2. Multiply a row by a *nonzero* number.

3. Add a constant multiple of one row to another.

You'll notice that these operations correspond exactly to the operations you can perform on a system of equations. Of course you can list the equation in any order, which corresponds to 1. You can multiply both sides of an equation by any nonzero number, which corresponds to 2. (We can't multiply both sides of an equation by 0, because although this preserves equality, we lose the equation.) And finally, 3. corresponds to the elimination method of solving equations.

Remark. If your are using the third row operation, whichever row you are using to change the others must remain the same from one step to the next.

This process isn't difficult, but it is easy to make computational mistakes. So if you get stuck or come up with an incorrect solution, it's usually best just to start over instead of trying to find your mistake.

Example 2. Solve the following system of equations.

$$x + y - 3z = 21$$
$$x = -7$$
$$-2x - 3z = 41$$

Solution. We perform the following elementary row operations.

$$
\left[\begin{array}{ccc|c}
1 & 1 & -3 & 21 \\
1 & 0 & 0 & -7 \\
-2 & 0 & -3 & 41
\end{array}\right]
\xrightarrow{R_1 \leftrightarrow R_2}
\left[\begin{array}{ccc|c}
1 & 0 & 0 & -7 \\
1 & 1 & -3 & 21 \\
-2 & 0 & -3 & 41
\end{array}\right]
\xrightarrow[2R_1+R_2]{-R_1+R_2}
\left[\begin{array}{ccc|c}
1 & 0 & 0 & -7 \\
0 & 1 & -3 & 28 \\
0 & 0 & -3 & 27
\end{array}\right]
$$

$$
\xrightarrow{-\frac{1}{3}R_3}
\left[\begin{array}{ccc|c}
1 & 0 & 0 & -7 \\
0 & 1 & -3 & 28 \\
0 & 0 & 1 & -9
\end{array}\right]
$$

Translating this back into equations, we get $z = -9$ from the third row. In the second row we have $y - 3z = 28$, but we know $z = -9$. So plugging this in, we find $y = 1$. And the first row reads $x = -7$. □

On solving word problems

Solving word problems can be overwhelming, and it can be difficult to figure out where to start. So here is an outline of one strategy in solving such problems.

Step 1: Read the problem; write down what you know and what you want to know.

Step 2: Determine what your variables are. Use letters that have significance to what they represent.

Step 3: Translate what you know into equations with the variables.

Step 4: Solve the system of equations as you normally would.

Step 5: Interpret what your solutions mean in the context of the problem.

Example 3. A goldsmith creates an 100-gram sample of a gold alloy which is 68.1% pure gold by mixing x grams of an alloy that is 78% pure gold and y grams of a second alloy that is 60% pure gold. How many grams does of the 78% pure gold alloy does he need?

Solution. We have x g of the 78% sample and y g of the 60% sample. We want a 100 g sample, so by conservation of mass we know x g and y g have to add up to 100 g. Percentages obviously aren't preserved, but the amount of gold in grams is. So if we have a 100-gram alloy which is 68.1% gold, then there are $100 \cdot .681 = 68.1$ grams of pure gold in the alloy. Moreover, there are $.78x$ grams of pure gold in the first alloy and $.6y$ grams of pure gold in the second. Putting all these facts together we can create a system of two equations in two unkowns.

$$x + y = 100 \tag{1}$$
$$.78x + .6y = 68.1 \tag{2}$$

We want to know how many grams of the 60%-pure gold alloy are needed. So we can solve (1) for y and plug this into (2) to have an equation in x. Doing this, we see $y = 100 - x$, so

$$.78x + .6(100 - x) = 68.1$$
$$.78x + 60 - .6x = 68.1$$
$$.18x = 8.1$$
$$x = 45$$

Since x was the number of grams of the 60%-pure gold alloy, this means we want 45 grams of this substance. □

Example 4. Two glasses of carrot juice and 4 glasses of milk together have a total of 716 mg of vitamin A while one glass of carrot juice and one glass of milk have a combined total of 221 mg of vitamin A. How much vitamin A is in a glass of carrot juice?

Solution. Let's call c the number of mg of vitamin A in one glass of carrot juice, and m the number of mg of vitamin A in one glass of milk. We want to know how much vitamin A is in one glass of carrot juice. Ultimately we want to know what c is. It should be clear from the given information that the equations we want are

$$2c + 4m = 716 \tag{3}$$
$$c + m = 221. \tag{4}$$

If we solve (4) for m, we can substitute this into (3) to obtain an equation in c. Clearly $m = 221 - c$, and making this substitution,

$$2c + 4(221 - c) = 716$$
$$2c + 884 - 4c = 716$$
$$168 = 2c$$
$$84 = c.$$

So there are 84 mg of vitamin A in one glass of carrot juice. If you're concerned about real-world accuracy, then this would line up if you have a 16-ounce glass of carrot juice. But there isn't near that much vitamin A in milk. □