Overview

In the previous lesson we discussed solving systems of linear equations and the method of Gaussian elimination. In this lesson we discuss the not-much-different method of Gauss-Jordan elimination.

Lesson

Recall that an augmented matrix is said to be in *row echelon form* if it looks something like the following.

$$\left[\begin{array}{ccc|c} 1 & * & * & a \\ 0 & 1 & * & b \\ 0 & 0 & 1 & c \end{array}\right] \tag{1}$$

We don't care what happens in the spots marked with a $*$, but what we want is a 1 in the first entry of each row, and all 0's below every leading 1. Note that we could have the last row being all 0s; that would be okay.

If we really wanted to, we could continue to perform the elementary row operations discussed in the previous lesson on the above matrix to further obtain a matrix of the form

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & a' \\ 0 & 1 & 0 & b' \\ 0 & 0 & 1 & c' \end{array}\right] \tag{2}$$

This type of matrix is said to be in *reduced row echelon form.* Note that if the matrix in (1) had a row of 0s, then the the matrix in (2) would have a row of 0s as well. The algorithm for obtaining a matrix in reduced row echelon form is called Gauss-Jordan elimination, and it differs only slightly from the method of Gaussian elimination learned in the previous lesson.

Gauss-Jordan elimination

1. Get a 1 in the top left entry.
2. Make every entry below this 1 a 0.
3. Go to the next row and find the first nonzero entry. Make this a 1.
4. Make every entry above and below this 1 a 0.
5. Repeat this process until you run out of rows.
6. Read off the solutions from the matrix by recalling which variable corresponds to which column.

The key difference in this algorithm is step 4. As we proceed through the algorithm, we want to make every thing *above* and *below* the leading 1s into 0s. Since this lesson isn't really all that different from what we did in the previous lesson, we only have two examples.

Example 1. Find the reduced row echelon form of the following matrix.

$$\left[\begin{array}{ccc|c} -2 & 1 & 3 & 32 \\ 1 & -1 & 0 & -5 \\ -2 & 0 & 2 & 26 \end{array}\right]$$

Solution.

$$\left[\begin{array}{ccc|c} -2 & 1 & 3 & 32 \\ 1 & -1 & 0 & -5 \\ -2 & 0 & 2 & 26 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{ccc|c} 1 & -1 & 0 & -5 \\ -2 & 1 & 3 & 32 \\ -2 & 0 & 2 & 26 \end{array}\right] \xrightarrow[2R_1+R_2]{2R_1+R_3} \left[\begin{array}{ccc|c} 1 & -1 & 0 & -5 \\ 0 & -1 & 3 & 22 \\ 0 & -2 & 2 & 16 \end{array}\right]$$

$$\xrightarrow{-1\cdot R_2} \left[\begin{array}{ccc|c} 1 & -1 & 0 & -5 \\ 0 & 1 & -3 & -22 \\ 0 & -2 & 2 & 16 \end{array}\right] \xrightarrow[R_1+R_2]{2R_1+R_3} \left[\begin{array}{ccc|c} 1 & 0 & -3 & -27 \\ 0 & 1 & -3 & -22 \\ 0 & 0 & -4 & -28 \end{array}\right]$$

$$\xrightarrow{-\frac{1}{4}R_3} \left[\begin{array}{ccc|c} 1 & 0 & -3 & -27 \\ 0 & 1 & -3 & -22 \\ 0 & 0 & 1 & 7 \end{array}\right] \xrightarrow[3R_3+R_1]{3R_3+R_2} \left[\begin{array}{ccc|c} 1 & 0 & 0 & -6 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 7 \end{array}\right]$$

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Example 2. Find the reduced row echelon form of the following matrix.

$$\left[\begin{array}{ccc|c} 1 & -5 & -1 & 33 \\ -2 & 2 & -4 & -20 \\ -5 & 2 & -2 & -43 \end{array}\right]$$

Solution.

$$\left[\begin{array}{ccc|c} 1 & -5 & -1 & 33 \\ -2 & 2 & -4 & -20 \\ -5 & 2 & -2 & -43 \end{array}\right] \xrightarrow[2R_2+R_2]{5R_1+R_3} \left[\begin{array}{ccc|c} 1 & -5 & -1 & 33 \\ 0 & -8 & -6 & 46 \\ 0 & -23 & -7 & 122 \end{array}\right] \xrightarrow{-3R_2+R_3} \left[\begin{array}{ccc|c} 1 & -5 & -1 & 33 \\ 0 & -8 & -6 & 46 \\ 0 & 1 & 11 & -16 \end{array}\right]$$

$$\xrightarrow{R_2 \leftrightarrow R_3} \left[\begin{array}{ccc|c} 1 & -5 & -1 & 33 \\ 0 & 1 & 11 & -16 \\ 0 & -8 & -6 & 46 \end{array}\right] \xrightarrow[8R_2+R_3]{5R_2+R_1} \left[\begin{array}{ccc|c} 1 & 0 & 54 & -47 \\ 0 & 1 & 11 & -16 \\ 0 & 0 & 82 & -82 \end{array}\right]$$

$$\xrightarrow{\frac{1}{82}R_3} \left[\begin{array}{ccc|c} 1 & 0 & 54 & -47 \\ 0 & 1 & 11 & -16 \\ 0 & 0 & 1 & -1 \end{array}\right] \xrightarrow[-11R_3+R_2]{-54R_3+R_1} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 7 \\ 0 & 1 & 0 & -5 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

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