Overview

In this lesson we cover determinants of square matrices. We'll see one valuable application of determinants in the next lesson. Again we only worry about 2×2 and 3×3 matrices and treat them separately.

Lesson

In general, given a square matrix A, we denote the determinant of A by $\det A$. Unlike inverses, we can compute the determinant of any square matrix. We can also denote the determinant of an actual matrix by replacing the brackets with vertical bars. For example if $A = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$, then

$$\det A = \begin{vmatrix} a & b \\ c & d \end{vmatrix}.$$

2×2 Determinants

The 2×2 case is really easy. Given a matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

we have $\det A = ad - bc$.

> **Remark.** This should remind you have how we computed the inverse of a 2×2 matrix. We could restate the inverse of A as
>
> $$A^{-1} = \frac{1}{\det A} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

Example 1. Compute the determinant of $A = \left[\begin{smallmatrix} 8 & 1 \\ 6 & -10 \end{smallmatrix}\right]$.

Solution. Just using the formula, $\det A = 8 \cdot -10 - 1 \cdot 6 = -86$. □

3×3 Determinants

Just as with finding inverses, the 2×2 case was an easy exception. In general, we need to use what is called *cofactor expansion* or expansion by minors.

> **Definition.** Given a matrix A, the *minor* M_{ij} is the determinant of the submatrix obtained from deleting the ith row and jth column of A. The *cofactor* c_{ij} is obtained by finding $(-1)^{i+j} M_{ij}$.

Example 2. Given the matrix

$$\begin{bmatrix} -4 & 4 & 2 \\ 1 & 1 & 0 \\ -3 & -5 & 2 \end{bmatrix},$$

find M_{32}, c_{32}, M_{11} and c_{11}.

Solution. Deleting the 3rd row and 2nd column, we have

$$M_{32} = \begin{vmatrix} -4 & 4 & 2 \\ 1 & 1 & 0 \\ -3 & -5 & 2 \end{vmatrix} = \begin{vmatrix} -4 & 2 \\ 1 & 0 \end{vmatrix} = -4 \cdot 0 - 2 \cdot 1 = -2.$$

This makes $c_{32} = (-1)^{3+2}(-2) = 2$. Now, deleting the first row and the first column,

$$M_{11} = \begin{vmatrix} -4 & 4 & 2 \\ 1 & 1 & 0 \\ -3 & -5 & 2 \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ -5 & 2 \end{vmatrix} = 1 \cdot 2 - 0 \cdot -5 = 2.$$

And this makes $c_{11} = (-1)^{1+1}(2) = 2$. □

There is an easier way to remember how to determine the cofactors of a matrix if you can remember the following "checkerboard."

$$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

Using the above checkerboard, we can now describe how to compute the determinant of a $3 \times 3 matrix$. Given a matrix

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix},$$

we can expand down any row or column using the checkerboard to compute $\det A$. For example, expanding down the first column of A, we have

$$\det A = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a \begin{vmatrix} e & f \\ h & i \end{vmatrix} - d \begin{vmatrix} b & c \\ h & i \end{vmatrix} + g \begin{vmatrix} b & c \\ e & f \end{vmatrix}$$

There is a precise formula that can succinctly describe this process in general, but hopefully the idea is clear. At this point we have reduced the 3×3 determinant to computing three 2×2 determinants, which we already have a formula for.

Example 3. Compute the determinant of

$$A = \begin{bmatrix} -6 & 5 & -2 \\ 5 & -3 & -3 \\ -5 & -2 & 4 \end{bmatrix}.$$

Solution. Expanding down the first column, we have

$$\begin{aligned} \det A &= -6 \begin{vmatrix} -3 & -3 \\ -2 & 4 \end{vmatrix} - 5 \begin{vmatrix} 5 & -3 \\ -5 & 4 \end{vmatrix} + (-2) \begin{vmatrix} 5 & -3 \\ -5 & -2 \end{vmatrix} \\ &= -6\left[(-3)(4) - (-3)(-2)\right] - 5\left[(5)(4) - (-5)(-3)\right] - 2\left[(5)(-2) - (-5)(-3)\right] \\ &= 133 \end{aligned}$$

□

Sometimes it's advantageous to pick out a particular row or column instead of arbitrarily choosing the first column, as the next example shows.

Example 4. Compute the determinant of

$$B = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 0 & 0 \\ 7 & -9 & 2 \end{bmatrix}.$$

Solution. If we choose to expand down the second row, two of the factors will be 0. So,

$$\det B = \begin{vmatrix} 1 & 2 & 3 \\ 1 & 0 & 0 \\ 7 & -9 & 2 \end{vmatrix} = -1 \cdot \begin{vmatrix} 2 & 3 \\ -9 & 2 \end{vmatrix} = -1\left[4 - (-27)\right] = -31. \qquad \square$$

> **Remark.** Why -1? Since we chose to start in the second row, the pattern from the checkerboard goes $-, +, -$.

Short cut for 3×3 determinants

While you will have to know how to find minors and cofactors for the homework and exams, there is a quicker and easier way to compute determinants in the case of a 3×3 matrix. The following trick works only for 3×3 matrices, but they come up often in this class, so it is worth knowing.

Say we want to compute the determinant of

$$A = \begin{bmatrix} 3 & -2 & -1 \\ -2 & -2 & 0 \\ -4 & 4 & 5 \end{bmatrix}.$$

We start by rewriting the first two columns and tracing out these diagonals.

$$\left|\begin{array}{ccc|cc} 3 & -2 & -1 & 3 & -2 \\ -2 & -2 & 0 & -2 & -2 \\ -4 & 4 & 5 & -4 & 4 \end{array}\right.$$

We multiply down the diagonals; we add the blue ones and subtract the red ones. So in this case, we have

$$\begin{aligned} \det A &= {\color{blue}(3)(-2)(5) + (-2)(0)(-4) + (-1)(-2)(4)} \\ &\quad {\color{red}- (-4)(-2)(-1) - (4)(0)(3) - (5)(-2)(-2)} \\ &= -30 + 8 + 8 - 20 \\ &= -34. \end{aligned}$$

Matrices with variables

In the coming lessons we'll want to be able to solve an equation like $\det A = 0$, where the entries of A have variables. We do an example like this now.

Example 5. Solve for x where

$$\begin{vmatrix} x-1 & -2 \\ -4 & x-3 \end{vmatrix} = 0.$$

Solution. Using the 2×2 formula we know, we get

$$\begin{aligned}
(x-1)(x-3)-(-4)(-2) &= 0 \\
x^2 - 4x + 3 - 8 &= 0 \\
x^2 - 4x - 5 &= 0 \\
(x+1)(x-5) &= 0,
\end{aligned}$$

giving $x = -1, 5$ as solutions. □

Finally, we end with a new definition and a fun fact. We say that a square matrix A is *singular* whenever $\det A = 0$. Otherwise, we say A is *nonsingular*.

> **Fun Fact.** A square matrix A is invertible if and only if $\det A \neq 0$. That is, A is invertible if and only if A is nonsingular.

This fun fact allows us to quickly determine if we can compute the inverse of a matrix since computing the inverse takes a bit more time.