Overview

This lesson covers the only application of the determinant that we care about in this class; namely finding eigenvalues and eigenvectors. Here we stick to the 2×2 case, and in the following lesson we will treat the 3×3 case.

Lesson

Although we will only do computations in the 2×2 case, we will give our definitions for any $n \times n$.

Eigenvalues and eigenvectors

Before we can define what an eigenvector is, we should clarify what we mean by vector. An n-dimensional (column) vector is just an $n \times 1$ matrix. We could also talk about row vectors, but we'll stick with column vectors, so the word "column" will generally be omitted.

Now suppose we have $A_{n\times n}$ and v is an n-dimensional vector. If λ is a nonzero real number such that

$$Av = \lambda v, \tag{1}$$

then λ is said to be an *eigenvalue* of A and v is the corresponding *eigenvector*. This means that multiplying A by v just returns some multiple of v.

Finding eigenvalues

Our goal is to find what values λ are eigenvalues for a given A. We can rearrange (1) to obtain

$$\begin{aligned} Av &= \lambda v \\ 0 &= \lambda v - Av \\ 0 &= (\lambda I - A) \end{aligned} \tag{2}$$

In the last line we factored out v. Since this is a matrix equation, we are left with λI instead of just λ. (Recall that if we did the same thing with real numbers, we would have $\lambda \cdot 1$. The only difference is we can't drop the I.)

Obviously, if v is the zero vector (just a vector of all 0s), then (2) is satisfied. We want to find nontrivial solutions to (2), so we ignore this case. Notice that $\lambda I - A$ is a matrix. As we discussed in a previous lesson, if $\lambda I - A$ is invertible, we can multiply both sides of the equation by $(\lambda I - A)^{-1}$. But this gives $v = (\lambda I - A)^{-1} = 0$, which is precisely what we said we wanted to avoid.

This means that if we are going to find any nontrivial solutions to (2), it must be that $\lambda I - A$ is not invertible. As we discovered in the previous lesson, this happens exactly when $\det(\lambda I - A) = 0$. It turns out that $\det(tI - A)$ is always a polynomial of degree n, and it is called the *characteristic polynomial* of A.

To find the eigenvalues for A, we find all solutions $t = \lambda$ to the polynomial equation

$$\det(tI - A) = 0.$$

Remark. As a convention, we will reserve λ to be solutions to the equation $\det(tI - A) = 0$, and the t will serve as the variable. As there will often by more than one solution, we'll mark different solutions with subscripts, e.g. λ_1, λ_2.

Example 1. Find the eigenvalues of the matrix $A = \begin{bmatrix} -4 & -2 \\ 10 & 8 \end{bmatrix}$.

Solution. We start by computing the characteristic polynomial of A.

$$\begin{aligned}
\det(tI - A) &= \left| \begin{bmatrix} t & 0 \\ 0 & t \end{bmatrix} - \begin{bmatrix} -4 & -2 \\ 10 & 8 \end{bmatrix} \right| \\
&= \begin{vmatrix} t+4 & 2 \\ -10 & t-8 \end{vmatrix} \\
&= (t+4)(t-8) - (2)(-10) \\
&= t^2 - 4t - 32 + 20 \\
&= t^2 - 4t - 12 \\
&= (t+2)(t-6)
\end{aligned}$$

Now we set $\det(tI - A) = (t+2)(t-6) = 0$. This gives solutions of $\lambda_1 = -2$ and $\lambda_2 = 6$. □

Example 2. Find the eigenvalues of the matrix $A = \begin{bmatrix} -19 & 40 \\ -16 & 33 \end{bmatrix}$.

Solution. Again starting with the characteristic polynomial of A,

$$\begin{aligned}
\det(tI - A) &= \left| \begin{bmatrix} t & 0 \\ 0 & t \end{bmatrix} - \begin{bmatrix} -19 & 40 \\ -16 & 33 \end{bmatrix} \right| \\
&= \begin{vmatrix} t+19 & -40 \\ 16 & t-33 \end{vmatrix} \\
&= (t+19)(t-33) + 640 \\
&= t^2 - 14t + 13 \\
&= (t-1)(t-13)
\end{aligned}$$

We immediately see that we have eigenvalues $\lambda_1 = 1$ and $\lambda_2 = 13$. □

Finding eigenvectors

One style of question to ask about eigenvectors is just to verify whether a given set of vectors has any eigenvectors. We do such an example.

Example 3. Which of the following are eigenvectors of $\begin{bmatrix} -7 & 6 \\ -4 & 4 \end{bmatrix}$?

$$\begin{bmatrix} 3 \\ 3 \end{bmatrix}, \quad \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 6 \\ 5 \end{bmatrix}, \quad \begin{bmatrix} -3 \\ -4 \end{bmatrix}$$

Solution. Such a question is really easy to answer. We just need to test whether equation (1) holds for each of the vectors above.

$$\begin{bmatrix} -7 & 6 \\ -4 & 4 \end{bmatrix} \begin{bmatrix} 3 \\ 3 \end{bmatrix} = \begin{bmatrix} -3 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} -7 & 6 \\ -4 & 4 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -8 \\ -4 \end{bmatrix} = -4 \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$
$$\begin{bmatrix} -7 & 6 \\ -4 & 4 \end{bmatrix} \begin{bmatrix} 6 \\ 5 \end{bmatrix} = \begin{bmatrix} -12 \\ -4 \end{bmatrix}$$
$$\begin{bmatrix} -7 & 6 \\ -4 & 4 \end{bmatrix} \begin{bmatrix} -3 \\ -4 \end{bmatrix} = \begin{bmatrix} -3 \\ -4 \end{bmatrix}$$

It's rather easy to see that the first and third vectors are not eigenvectors. For the second one, we can pull out a factor of -4 to see that $\left[\begin{smallmatrix} 2 \\ 1 \end{smallmatrix}\right]$ is an eigenvector with eigenvalue -4, and clearly $\left[\begin{smallmatrix} -3 \\ -4 \end{smallmatrix}\right]$ is an eigenvector with eigenvalue 1. □

If we want to find the eigenvectors on our own, it involves solving a simple system of equations. In order to find the eigenvectors of a matrix, we must first find the eigenvalues.

Example 4. Find the eigenvalues and the corresponding eigenvectors of the matrix

$$A = \begin{bmatrix} 0 & -2 \\ 5 & -7 \end{bmatrix}.$$

Solution. We start as usual.

$$\begin{aligned} \det(tI - A) &= \left| \begin{bmatrix} t & 0 \\ 0 & t \end{bmatrix} - \begin{bmatrix} 0 & -2 \\ 5 & -7 \end{bmatrix} \right| \\ &= \begin{vmatrix} t & 2 \\ -5 & t+7 \end{vmatrix} \\ &= t(t+7) + 10 \\ &= t^2 + 7t + 10 \\ &= (t+2)(t+5) \end{aligned}$$

This gives us $\lambda_1 = -2$ and $\lambda_2 = -5$ as our eigenvalues. Recall now what it means to be an eigenvector and eigenvalue. Let's say that $v_1 = \left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right]$ is the corresponding eigenvector to λ_1. Then

$$(\lambda_1 I - A)v_1 = 0$$

That is, we want to solve the matrix equation

$$\underbrace{\begin{bmatrix} -2 & 2 \\ -5 & 5 \end{bmatrix}}_{\lambda_1 I - A} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

where all we've done is plugged $\lambda_1 = -2$ into $\lambda_1 I - A$. Putting this equation into an augmented matrix, we have

$$\left[\begin{array}{cc|c} -2 & 2 & 0 \\ -5 & 5 & 0 \end{array}\right]$$

Now row reducing, we get

$$\left[\begin{array}{cc|c} -2 & 2 & 0 \\ -5 & 5 & 0 \end{array}\right] \xrightarrow{-\frac{1}{2}R_1} \left[\begin{array}{cc|c} 1 & -1 & 0 \\ -5 & 5 & 0 \end{array}\right] \xrightarrow{5R_1+R_2} \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Since the column corresponding to y does not have a leading 1, it is a *free variable*. Set $y = t$. then the first row tells us that $x - t = 0$, or $x = t$, so we have $v_1 = \left[\begin{smallmatrix} t \\ t \end{smallmatrix}\right]$. But this is really infinitely many vectors (for any choice of t). So we may pick any t that we wish, say $t = 1$. Then $\lambda_1 = -2$ has the corresponding eigenvector $v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

We treat $\lambda_2 = -5$ similarly. Plugging this into $[tI - A \mid 0]$, we get

$$\left[\begin{array}{cc|c} -5 & 2 & 0 \\ -5 & 2 & 0 \end{array}\right] \xrightarrow{-R_1+R_2} \left[\begin{array}{cc|c} -5 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Again we are free to pick $y = t$. then the first row says $-5x + 2t = 0$, or $x = \frac{2}{5}t$. So picking $t = 2$ gives $v_2 = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$ corresponding to $\lambda_2 = -5$. □

> **Remark.** There is nothing special about our choice of t. If you wanted, you could choose $t = 1$ every time. Loncapa will accept any scalar multiple of v_1 and v_2 as they would still satisfy the equation in (1).

Example 5. Find the eigenvalues and corresponding eigenvectors for the matrix

$$A = \begin{bmatrix} -32 & 8 \\ -4 & 1 \end{bmatrix}.$$

Solution. We compute the characteristic polynomial of A.

$$\begin{aligned} \det(tI - A) &= \left| \begin{bmatrix} t & 0 \\ 0 & t \end{bmatrix} - \begin{bmatrix} -32 & 8 \\ -4 & 1 \end{bmatrix} \right| \\ &= \begin{vmatrix} t+32 & -8 \\ 4 & t-1 \end{vmatrix} \\ &= (t+32)(t-1) + 32 \\ &= t^2 + 31t \end{aligned}$$

This gives us eigenvalues of $\lambda_1 = 0$ and $\lambda_2 = -31$. Here we will illustrate how we could solve the system of equations without using an augmented matrix. Finding v_1, we have

$$\begin{cases} 32x - 8y &= 0 \\ 4x - y &= 0 \end{cases}$$

Using the second equation, we have $y = 4x$. Substituting this into the first equation gives $32x - 8(4x) = 32x - 32x = 0$, which just tells us $0 = 0$. This means we can pick $x = t$, then $y = 4t$. So let's pick $t = 1$ so $v_1 = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$ is the eigenvector for $\lambda_1 = 0$. Similarly finding v_2 we have the system of equations

$$\begin{cases} x - 8y &= 0 \\ 4x - 32y &= 0 \end{cases}$$

We see that the second equation is a multiple of the first equation (we could have observed this when finding v_1 as well). The first equation tells us $x = 8y$, so we are free to pick $y = 1$, then $x = 8$. So $v_2 = \begin{bmatrix} 8 \\ 1 \end{bmatrix}$ corresponds to eigenvalue $\lambda_2 = -31$. $\square$