Overview

As discussed in the previous lesson, we find eigenvalues and eigenvectors, now restricting our attention to 3×3 matrices. There really isn't any new information for this lesson, so we jump straight to the examples.

Examples

Example 1. Find the eigenvalues of

$$A = \begin{bmatrix} -11 & 4 & 8 \\ -10 & 3 & 8 \\ -6 & 2 & 5 \end{bmatrix}$$

Solution. To start, we find

$$tI - A = \begin{bmatrix} t & 0 & 0 \\ 0 & t & 0 \\ 0 & 0 & t \end{bmatrix} - \begin{bmatrix} -11 & 4 & 8 \\ -10 & 3 & 8 \\ -6 & 2 & 5 \end{bmatrix} = \begin{bmatrix} t+11 & -4 & -8 \\ 10 & t-3 & -8 \\ 6 & -2 & t-5 \end{bmatrix}$$

This is where using our 3×3 determinant trick is particularly helpful. To compute $\det(tI-A)$, we write

$$\left|\begin{array}{ccc|cc} t+11 & -4 & -8 & t+11 & -4 \\ 10 & t-3 & -8 & 10 & t-3 \\ 6 & -2 & t-5 & 6 & -2 \end{array}\right.$$

Recall that we add the forward diagonals and subtract the backward diagonals. Then

$$\begin{aligned} \det(tI-A) &= (t+11)(t-3)(t-5) + (-4)(-8)(6) + (-8)(10)(-2) \\ &\quad - (-8)(t-3)(6) - (-2)(-8)(t+11) - (t-5)(10)(-4) \\ &= (t^2+8t-33)(t-5) + 192 + 160 + 48(t-3) - 16(t+11) + 40(t-5) \\ &= t^3 + 3t^2 - 73t + 165 + 192 + 160 + 48t - 144 - 16t - 176 + 40t - 200 \\ &= t^3 + 3t^2 - t - 3 \end{aligned}$$

Now we need to find solutions to $t^3 + 3t^2 - t - 3 = 0$. One way to do this is by grouping.

$$\begin{aligned} t^3 + 3t^2 - t - 3 &= 0 \\ t^3 - t + 3t^2 - 3 &= 0 \\ t(t^2-1) + 3(t^2-1) &= 0 \\ (t+3)(t^2-1) &= 0 \\ (t+3)(t+1)(t-1) &= 0 \end{aligned}$$

So we have eigenvalues $\lambda_1 = 1, \lambda_2 = -1, \lambda_3 = -3$. □

Example 2. Compute the eigenvalues of the matrix

$$A = \begin{bmatrix} -11 & 2 & 10 \\ -8 & -1 & 10 \\ -6 & 1 & 6 \end{bmatrix}$$

Solution. To compute $\det(tI - A)$, we again use our trick

$$\left|\begin{matrix} t+11 & -2 & -10 \\ 8 & t+1 & -10 \\ 6 & -1 & t-6 \end{matrix}\right|\begin{matrix} t+11 & -2 \\ 8 & t+1 \\ 6 & -1 \end{matrix}$$

Now

$$\begin{aligned}\det(tI - A) &= (t+11)(t+1)(t-6) + (-2)(-10)(6) + (-10)(8)(-1) \\ &\quad - (6)(t+1)(-10) - (-1)(-10)(t+11) - (t-6)(8)(-2) \\ &= (t^2 + 12t + 11)(t-6) + 120 + 80 + 60t + 60 - 10t - 110 + 16t - 96 \\ &= t^3 + 6t^2 - 61t - 66 + 120 + 80 + 60t + 60 - 10t - 110 + 16t - 96 \\ &= t^3 + 6t^2 + 5t - 12\end{aligned}$$

We use another method to find the roots of the polynomial $f(t) = t^3 + 6t^2 + 5t - 12$. While there isn't an easy-to-memorize version of the quadratic formula for cubics, we can use some strategic guessing to figure out the roots of the polynomial. We know that their product must be equal to -12. The factors of -12 are $\pm 1, 2, 3, 4, 6$. To find a zero, simply just start plugging the numbers into f. We can use the intermediate value theorem to aid our guessing.

For example $f(-1) = -12$, while $f(2) = 30$. This means there must be a zero between -1 and 2. The only factor -12 on this interval is 1. And sure enough $f(1) = 0$. What do we do now? We need to divide $f(t)$ by $t - 1$. We can either use polynomial long division or synthetic division. We will demonstrate synthetic division here.

Recall that we put the root on the left, the coefficients along the top (leaving 0s as placeholders if needed). Then we add down and multiply up. This is meant to be a quick refresh. If you need more details, see for example `Purple Math`.

$$\begin{array}{r|rrrr} & 1 & 6 & 5 & -12 \\ 1 & & 1 & 7 & 12 \\ \hline & 1 & 7 & 12 & 0 \end{array}$$

This means that $f(t) = (t-1)(t^2 + 7t + 12)$. Now we can more easily factor the quadratic to obtain that $f(t) = (t-1)(t+3)(t+4)$. This means that we have $\lambda_1 = 1$, $\lambda_2 = -3$ and $\lambda_3 = -4$ as our eigenvalues. □

Example 3. Compute the eigenvalues and eigenvectors of

$$A = \begin{bmatrix} 3 & -1 & -2 \\ 0 & 2 & 18 \\ 0 & 1 & -1 \end{bmatrix}$$

Solution. We have

$$tI - A = \begin{bmatrix} t-3 & 1 & 2 \\ 0 & t-2 & -18 \\ 0 & -1 & t+1 \end{bmatrix}$$

In this example, it is probably quicker to use expansion by minors down the first column to compute $\det(tI - A)$. We have

$$\begin{aligned} f(t) &= \begin{vmatrix} t-3 & 1 & 2 \\ 0 & t-2 & -18 \\ 0 & -1 & t+1 \end{vmatrix} \\ &= (t-3)\begin{vmatrix} t-2 & -18 \\ -1 & t+1 \end{vmatrix} \\ &= (t-3)\left[(t-2)(t+1) - (-1)(-18)\right] \\ &= (t-3)(t-2)(t+1) - 18(t-3) \\ &= (t^2 - 5t + 6)(t+1) - 18t + 18 \\ &= t^3 - 4t^2 + t + 6 - 18t + 54 \\ &= t^3 - 4t^2 - 17t + 60. \end{aligned}$$

We list the factors of $60 : \pm 1, 2, 3, 4, 5, 6, 10, 12, 15, 20, 30, 60$. There are 24 possibilities here. But factors closest to the center are most likely. If we try 6 we get $f(6) = 30$, which is too big, but we find that $f(5) = 0$. So $t - 5$ divides $f(t)$. Using synthetic division, we get

$$\begin{array}{r|rrrr} & 1 & -4 & -17 & 60 \\ 5 & & 5 & 5 & -60 \\ \hline & 1 & 1 & -12 & 0 \end{array}$$

So we have $f(t) = (t-5)(t^2 + t - 12) = (t-5)(t+4)(t-3)$, giving $\lambda_1 = 5$, $\lambda_2 = -4$, $\lambda_3 = 3$. Now we have to find the corresponding eigenvectors. We use the same matrix method we learned in the previous lesson. For $\lambda_1 = 5$, we plug 5 in for t in $tI - A$ and row reduce. We are looking for $v_1 = \left[\begin{smallmatrix} x \\ y \\ z \end{smallmatrix}\right]$

$$\left[\begin{array}{ccc|c} 2 & 1 & 2 & 0 \\ 0 & 3 & -18 & 0 \\ 0 & -1 & 6 & 0 \end{array}\right] \xrightarrow{\frac{1}{2}R_1} \left[\begin{array}{ccc|c} 1 & \frac{1}{2} & 1 & 0 \\ 0 & 3 & -18 & 0 \\ 0 & -1 & 6 & 0 \end{array}\right]$$

$$\xrightarrow{R_2 \leftrightarrow R_3} \left[\begin{array}{ccc|c} 1 & \frac{1}{2} & 1 & 0 \\ 0 & -1 & 6 & 0 \\ 0 & 3 & -18 & 0 \end{array}\right] \xrightarrow{-R_2} \left[\begin{array}{ccc|c} 1 & \frac{1}{2} & 1 & 0 \\ 0 & 1 & -6 & 0 \\ 0 & 3 & -18 & 0 \end{array}\right]$$

$$\xrightarrow{-3R_2 + R_3} \left[\begin{array}{ccc|c} 1 & \frac{1}{2} & 1 & 0 \\ 0 & 1 & -6 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

This means that z is the free variable (column 3 has no leading 1). We pick $z = 1$ (because we can). Then the second row says that $y - 6 = 0$, or $y = 6$. Finally the first row says $x + \frac{1}{2}(6) + 1 = 0$, or $x = -4$. So $v_1 = \left[\begin{smallmatrix} -4 \\ 6 \\ 1 \end{smallmatrix}\right]$.

We rinse and repeat to find v_2 and v_3. Since the last column is always 0, we don't really need to write it.

$$\begin{bmatrix} -7 & 1 & 2 \\ 0 & -6 & -18 \\ 0 & -1 & -3 \end{bmatrix} \xrightarrow[-\frac{1}{6}R_2]{-\frac{1}{7}R_1} \begin{bmatrix} 1 & -\frac{1}{7} & -\frac{2}{7} \\ 0 & 1 & 3 \\ 0 & -1 & -3 \end{bmatrix} \xrightarrow{R_2 + R_3} \begin{bmatrix} 1 & -\frac{1}{7} & -\frac{2}{7} \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

In this case, we see that z is free again, so let's pick $z = 1$. Then the second row says that $y + 3 = 0$, which gives $y = -3$. And the first row tells us that $x - \frac{1}{7}(-3) - \frac{2}{7} = 0$. Solving for x gives $x = -\frac{1}{7}$. So we get $v_2 = \begin{bmatrix} -1/7 \\ -3 \\ 1 \end{bmatrix}$. We could multiply v_2 by 7 to get rid of the fractions to obtain $v_2 = \begin{bmatrix} 1 \\ 21 \\ -7 \end{bmatrix}$.

Finally we find v_3.

$$\begin{bmatrix} 0 & 1 & 2 \\ 0 & 1 & -18 \\ 0 & -1 & 4 \end{bmatrix} \xrightarrow[-R_1+R_2]{R_1+R_3} \begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & -20 \\ 0 & 0 & 6 \end{bmatrix}$$

Column 1 has no leading 1, so x is the free variable. The second two columns say that $z = 0$, and using the first row, we see that $y = 0$ as well. Picking $x = 1$, we have that $v_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$. $\square$