Overview

In this lesson we start our study of differential equations. We start by considering only exponential growth and decay, and in the next lesson we will extend this idea to the general method of separation of variables. An important application in this lesson is Newton's Law of Cooling.

Lesson

In order to talk about differential equations, we need to know what such a thing is. It turns out to be only slightly more complicated than what we are already familiar with.

Definition 1. A *differential equation* is an equation that relates a function with its derivatives.

If y is a function of t, an example of a differential equation would be

$$5y' + 3yt = 7t^2.$$

Notice that we can have factors of the independent variable t floating around here. Today and the next two lessons we are only concerned with *separable equations*.

Definition 2. A *separable equation* is a differential equation where we can get all the y's on one side and all the x's (or t's, whatever the independent variable is) on the other. The method of solving separable equations is called *separation of variables*.

To see that Definition 2 is only a minor extension of what we have learned thus far, consider something like

$$y = \int \frac{1}{t}\,dt.$$

Differentiating both sides, we get

$$\frac{d}{dt}\,y = \frac{d}{dt}\int \frac{1}{t}\,dt$$

$$\frac{dy}{dt} = \frac{1}{t}. \tag{1}$$

So (1) is a differential equation. To solve this, we work backward to get a solution of $y = \ln|t| + C$. The extension in this lesson is that we will have something more complicated than $y =$ something, so we will have to solve for y.

Example 1. Find the general solution for the differential equation

$$\frac{dy}{dx} = 14\frac{x^7 + 3}{y^2}.$$

Solution.

$$\begin{aligned}
y^2\frac{dy}{dx} &= 14(x^7+3)\\
y^2\,dy &= 14(x^7+3)\,dx\\
\int y^2\,dy &= \int 14(x^7+3)\,dx\\
\frac{1}{3}y^3 &= \frac{14}{8}x^8 + 42x + C\\
y^3 &= \frac{42}{8}x^8 + 126x + C_1\\
y &= (\frac{42}{8}x^8 + 126x + C_1)^{1/3}
\end{aligned}$$

where $C_1 = 3C$. □

Remark. For purposes of Loncapa, any time we modify C to get a new C_1, we will use the convention of relabeling C_1 as C. This may be confusing at first, but C is just a constant, and we don't care what it's actual value is.

Before moving on to the next example we need to recall what it means to be directly proportional.

Definition 3. If a, b, c are variables, then a is *directly proportional* to b means that there is a constant k such that $a = kb$. Similarly, a is *jointly proportional* to b and c if there is a constant k such that $a = kbc$.

Now we can apply what we have learned to exponential growth and decay. A reasonable model for growth of populations or decay of radioactive material is that the rate of growth (decay) is directly proportional to the amount at the given time. Say we have a radioactive material whose amount is given by the function $A(t)$. Then using the definition above, we have that

$$\frac{dA}{dt} = kA. \tag{2}$$

Example 2. Americium-241 is a ubiquitous isotope of Am, and is probably found in your household smoke detector. The half-life of ^{241}Am is 432.2 years. If your smoke detector has 4 micrograms of ^{241}Am when you move into your house, how much will remain when you pay off your 30-year mortgage?

Solution. Using (2) and the method of separation of variables,

$$\begin{aligned}
\frac{dA}{dt} &= kA \\
dA &= kA\,dt \\
\frac{dA}{A} &= k\,dt \\
\int \frac{dA}{A} &= \int k\,dt \\
\ln|A| &= kt + C
\end{aligned}$$

Since we can't have a negative amount of $^{241}\mathrm{Am}$, we can ignore the absolute values. Now solving for C, we have

$$\begin{aligned}
\ln A(0) = \ln 4 &= k \cdot 0 + C \\
\ln 4 &= C.
\end{aligned}$$

Next we want to solve for A to find the function $A(t)$.

$$\begin{aligned}
\ln A &= kt + \ln 4 \\
A &= e^{kt+\ln 4} \\
A &= e^{kt}e^{\ln 4} \\
A &= 4e^{kt}.
\end{aligned}$$

And using the fact that $A(432.2) = 2 = \frac{1}{2}A(0)$,

$$\begin{aligned}
\frac{1}{2} &= e^{k(432.2)} \\
\ln\frac{1}{2} &= k(432.2) \\
\frac{\ln\frac{1}{2}}{432.2} &= k \\
-0.0016038 &\approx k.
\end{aligned}$$

Putting this together, we find that $A(t) = 4e^{-0.0016038t}$. Then we're asked to find $A(30) = 4e^{-0.0016038\cdot 30} \approx 3.8\ \mu\mathrm{g}$. □

Examples 1 and 2 illustrate two types of solutions. In Example 1, we found a *general solution*, and in Example 2 we found a *particular solution*.

Definition 4. A *general solution* to a differential equation is an infinite number solutions accounting for the addition of an arbitrary constant C. A *particular solution* to a differential equation is a single solution, where we have determined C using the given conditions. An *initial value problem* is a differential equation where we find a particular solution given $y(t_0)$ (and perhaps $y'(t_0)$, $y''(t_0)$, etc.).

Example 3. Find a particular solution to the given differential equation.

$$\frac{dy}{dx} = 6x^2e^{5y-x^3}$$

Solution. Notice that

$$\frac{dy}{dx} = 6x^2 e^{5y-x^3} = 6x^2 e^{5y} e^{-x^2}, \qquad y(1) = 2.$$

So this equation is separable, and

$$\begin{aligned}
e^{-5y}\,dy &= 6x^2 e^{-x^3}\,dx \\
\int e^{-5y}\,dy &= \int 6x^2 e^{-x^3}\,dx \\
-\frac{1}{5}e^{-5y} &= \frac{6}{-3}\int e^u\,du && u = -x^3 \\
& && du = -3x^2\,dx \\
-\frac{1}{5}e^{-5y} &= -2e^u + C \\
-\frac{1}{5}e^{-5y} &= -2e^{-x^3} + C \\
-\frac{1}{5}e^{-5\cdot 2} &= -2e^{-1^3} + C \\
-\frac{1}{5}e^{-10} &= -2e^{-1} + C \\
C &= 2e^{-1} - \frac{1}{5}e^{-10}.
\end{aligned}$$

Now solving for y,

$$\begin{aligned}
-\frac{1}{5}e^{-5y} &= -2e^{-x^3} + 2e^{-1} - \frac{1}{5}e^{-10} \\
e^{-5y} &= 10e^{-x^3} - 10e^{-1} + e^{-10} \\
-5y &= \ln\left(10e^{-x^3} - 10e^{-1} + e^{-10}\right) \\
y &= -\frac{1}{5}\ln\left(10e^{-x^3} - 10e^{-1} + e^{-10}\right).
\end{aligned}$$

□

In the next example, we will revisit the spirit of Example 5 of Lesson 1 where we wanted to determine the time of death. With our new knowledge we can actually derive a formula like the one that was given in Lesson 1 using Newton's Law of Cooling.

Theorem (Newton's Law of Cooling)**.** Given an object whose temperature is a function of time, $T(t)$ whose surroundings are a constant temperature, the change in temperature of the object is directly proportional to the difference of the temperature at time t of the object and the ambient temperature.

Ambient temperature just means the temperature of the surroundings. If we represent this (constant) number with T_A, then Newton's Law of Cooling says

$$\frac{dT}{dt} = k(T - T_A). \tag{3}$$

Example 4. You arrive at a crime scene at 6:00 am and discover a body. Crime scene investigators measure the body's temperature to be 27°C upon arrival, and an hour later the body's temperature is 25°C. During this time, the temperature of the room was 22°C. Assuming that the person a temperature of 37°C when living, what was the time of death?

Solution. In this example, $T_A = 22$. By Newton's Law of Cooling (3), we have

$$\begin{aligned} \frac{dT}{dt} &= k(T-22) \\ \frac{dT}{T-22} &= k\,dt \\ \int \frac{dT}{T-22} &= \int k\,dt \\ \ln|T-22| &= kt + C \end{aligned} \tag{4}$$

Since the temperature of the body can't drop below the temperature of the room (because science), we can remove the absolute value bars. Letting 6:00 am be $t = 0$, we know that $T(0) = 27$ and $T(1) = 25$. We use the former to solve for C:

$$\ln(27-22) = 0 + C \Rightarrow C = \ln 5.$$

Now we can use $T(1) = 25$ to solve for k.

$$\begin{aligned} \ln(T-22) &= kt + \ln 5 \\ T - 22 &= e^{kt+\ln 5} \\ T &= 22 + 5\ln e^{kt} \\ T(1) = 25 &= 22 + 5\ln e^{k} \\ 3 &= 5e^{k} \\ \ln\frac{3}{5} &= k. \end{aligned}$$

Putting this together,

$$T(t) = 22 + 5e^{t\ln\frac{3}{5}} \tag{5}$$

At the time of death we know that $T = 37$, so we need to solve for t. To do this, we could use (5):

$$\begin{aligned} 37 &= 22 + 5e^{t\ln\frac{3}{5}} \\ 15 &= 5e^{t\ln\frac{3}{5}} \\ 3 &= e^{t\ln\frac{3}{5}} \\ \ln 3 &= t\ln\frac{3}{5} \qquad (6) \\ \frac{\ln 3}{\ln\frac{3}{5}} &= t \\ t &\approx -2.15\,\text{h} \\ &= -2\,\text{h}9\,\text{m}. \end{aligned}$$

Alternatively, we could use (4). This gives us

$$\begin{aligned}\ln(37-22) &= t\ln\frac{3}{5}+\ln 5\\ \ln 15-\ln 5 &= t\ln\frac{3}{5}\\ \ln 3 &= t\ln\frac{3}{5}.\end{aligned}$$

Now we're back at the same point as (6). So the algebra in the second method was just a little bit easier/quicker. In either case, we find thatn the time of death was 2 hours and 9 minutes ago, 3:51 am. □

> **Remark.** It turns out that the solution to any Newton's law of cooling problem has the form
> $$T(t) = T_A + Ce^{kt}.$$
> Note that the C here is actually e^{C_1}, where C_1 is the integration constant we found in the previous example.

Example 5. Find a particular solution to the differential equation (n is a constant)

$$y' = 6x^n, \qquad y(1) = 3.$$

Solution. We need to separate this problem into two cases: $n=-1$ and $n\neq -1$. If $n\neq -1$, then

$$\begin{aligned}\frac{dy}{dx} &= 6x^n\\ dy &= 6x^n\,dx\\ \int dy &= \int 6x^n\,dx \qquad (*)\\ y &= \frac{6}{n+1}x^{n+1}+C.\end{aligned}$$

Solving for C is straightforward, $y(1)=3$ should give $C=\frac{1}{2}$. So a particular solution in this case is

$$y = \frac{6}{n+1}x^{n+1}+C.$$

If $n=-1$, however, we can't use the power rule. Our work up to $(*)$ above remains the same. Now

$$\begin{aligned}\int dy &= \int 6x^-1\,dx\\ y &= 6\ln|x|+C.\end{aligned}$$

And $y(1)=3$ gives us $C=3$. So a particular solution in this case is

$$y = 6\ln|x|+3.$$ □