Overview

This is yet another lesson on separable equations, but the word problems are a bit more involved which make it a whole nother beast entirely. Tank problems are rather ubiquitous in differential equations. Usually we will have multiple things going on, like water flowing into and out of a tank. The key is

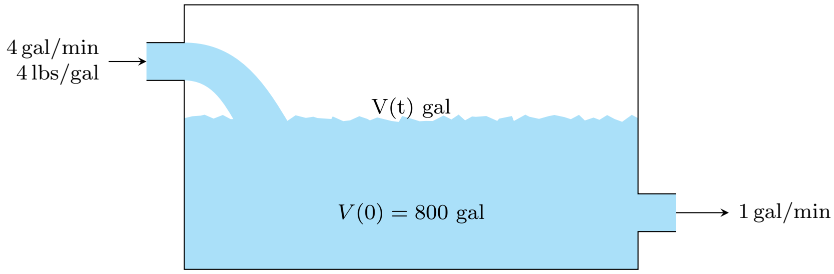
$$\text{total rate} = (\text{rate in}) - (\text{rate out}).$$

Keeping this in mind, let's look at some examples.

Examples

Example 1. A 1000-gallon tank initially contains 800 gallons of brine containing 75 pounds of dissolved salt. Brine containing 3 pounds of salt per gallon flows into the tank at the rate of 4 gallons per minute, and the well-stirred mixture flows out of the tank at the rate of 1 gallon per minute. Set up a differential equation for the amount of salt, $A(t)$, in the tank at time t.

Solution. Let's first consider the volume brine as a function of time, $V(t)$.



Notice that we have 4 gal/min flowing in and 1 gal/min flowing out. Thus

$$\frac{dV}{dt} = 4 - 1 = 3\ \frac{\text{gal}}{\text{min}}.$$

Now using the fact that $V(0) = 800$, we know that volume is given by

$$V(t) = 800 + 3t. \tag{1}$$

Setting that aside for a minute, we turn to the problem at hand. How much salt is in the brine at time t? Let $A(t)$ be the amount of salt in pounds at time t. We know that the concentration of what's coming in is 4 lbs/gal. Moreover, $\frac{A(t)}{V(t)}$ lbs/gal is the concentration of the brine that is leaving the tank at time t.

$$\begin{aligned}
\frac{dA}{dt} &= \frac{3\text{ lbs}}{\text{gal}} \cdot \frac{4\text{ gal}}{\text{min}} - \frac{A(t)\text{ lbs}}{V(t)\text{ gal}} \cdot \frac{1\text{ gal}}{\text{min}} \\
\frac{dA}{dt} &\overset{(1)}{=} \frac{3\text{ lbs}}{\text{gal}} \cdot \frac{4\text{ gal}}{\text{min}} - \frac{A(t)\text{ lbs}}{800+3t\text{ gal}} \cdot \frac{1\text{ gal}}{\text{min}} \\
&= \left[12 - \frac{A(t)}{800+3t}\right]\ \text{gal/min}.
\end{aligned}$$

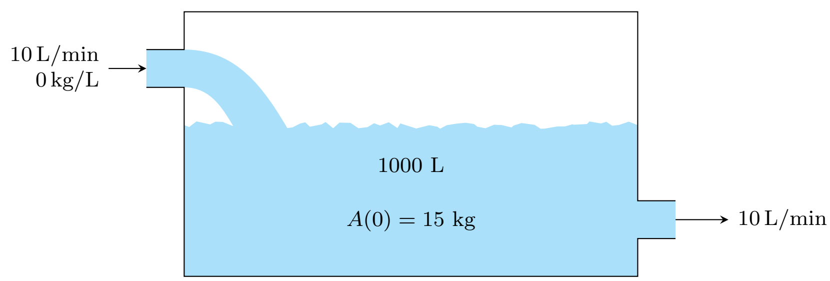
So the differential equation is simply

$$\frac{dA}{dt} = 12 - \frac{A}{800+3t}. \qquad \square$$

Remark. How did we know what to make $A(t)$? A good rule of thumb is to look at what specifically the question is asking, and then try to make your equations match that. For instance, in this example we were looking for the amount of salt in pounds at time t, so we made a function $A(t)$ which represented the amount of salt in pounds at time t.

Example 2. A tank contains 1000 L of brine with 15 kg of dissolved salt. Pure water enters the tank at a rate of 10 L/min. The solution is kept thoroughly mixed and drains from the tank at the same rate. How much salt is in the tank (a) after t minutes and (b) after 20 minutes?

Solution. Again we make $A(t)$ the amount of salt in kg at time t. The picture is like this.



Then

$$\begin{aligned}\frac{dA}{dt} &= \frac{0\text{ kg}}{\text{L}} \cdot \frac{10\text{ L}}{\text{min}} - \frac{A(t)\text{ kg}}{1000\text{ L}} \cdot \frac{10\text{ L}}{\text{min}}\\ &= -\frac{A(t)}{100}\,\frac{\text{kg}}{\text{min}}\end{aligned}$$

That is

$$\begin{aligned}\frac{dA}{dt} &= -\frac{A}{100}\\ \frac{dA}{A} &= -\frac{1}{100}\,dt\\ \int \frac{dA}{A} &= \int -\frac{1}{100}\,dt\\ \ln|A| &= -\frac{t}{100} + C_1\\ A &= e^{-t/100+C_1}\\ A &= Ce^{-t/100}.\end{aligned}$$

Using that $A(0) = 15$, we have

$$A(t) = 15e^{-t/100}.$$

This answers (a). And for (b), we are looking for $A(20) = 15e^{-.2} \approx 12.3$ kg. $\square$

Example 3. In a particular chemical reaction, a substance is converted into a second substance at a rate proportional to the square of the amount of the first substance present at time t. Initially, 42 grams of the first substance was present, and 1 hour later only 15 grams of the first substance remained. What is the amount of the first substance remaining after 3 hours?

Solution. It is easy to get lost in the mix here, but what is our goal? We want to find the *amount* in grams of the first substance remaining after a certain time. So let $A(t)$ represent the amount in grams of the first substance at time t. We are told that the rate of change of that amount is (directly) proportional to the square of the amount remaining. In a formula, this means

$$\frac{dA}{dt} = kA^2.$$

We are also given $A(0) = 42$ and $A(1) = 15$. From this point on, we just have to solve a regular initial value problem.

$$\begin{aligned}
\frac{dA}{dt} &= kA^2 \\
A^{-2}\,dA &= k\,dt \\
\int A^{-2}\,dA &= \int k\,dt \\
-A^{-1} &= kt + C \\
-\frac{1}{42} &= C.
\end{aligned}$$

In this last line we just plugged in 0 for t and 42 for A. Using $A(1) = 15$,

$$\begin{aligned}
-\frac{1}{A} &= kt - \frac{1}{42} \\
-\frac{1}{15} &= k - \frac{1}{42} \\
k &= \frac{1}{42} - \frac{1}{15} \\
k &= -\frac{3}{70}.
\end{aligned}$$

Putting it all together,

$$\begin{aligned}
-\frac{1}{A} &= -\frac{3}{70}t - \frac{1}{42} \\
\frac{1}{A} &= \frac{3}{70}t + \frac{1}{42} \\
A(t) &= \frac{1}{\frac{3}{70}t + \frac{1}{42}} \\
A(3) &= \frac{1}{\frac{3}{70}\cdot 3 + \frac{1}{42}} \\
&= \boxed{-6.5625}
\end{aligned}$$

□

Recall. a is said to be *inversely proportional* to b if there is some constant k such that $a = \frac{k}{b}$. One could also say that a is *directly proportional* to $\frac{1}{b}$.

Example 4. The rate of change in the number of miles of road cleared per hour by a snowplow is inversely proportional to the depth of the snow. Given that 21 miles per hour are cleared when the depth of the snow is 2.2 inches and 13 miles per hour are cleared when the depth of the snow is 8 inches, then how many miles of road will be cleared each hour when the depth of the snow is 11 inches?

Solution. The wording here is also kind of tricky. It should make sense that the function we want is the number of miles per hour cleared, call it N. But in this problem N doesn't depend on time, it depends on the depth of snow, say s in inches. So our function is $N(s)$.

Now the problem states that the rate of change of N is *inversely proportional* to s. That is,

$$\frac{dN}{ds} = \frac{k}{s}.$$

We are also given as initial conditions $N(2.2) = 21$ and $N(8) = 13$. Now

$$\begin{aligned} \frac{dN}{ds} &= \frac{k}{s} \\ dN &= \frac{k}{s}\,ds \\ \int dN &= \int \frac{k}{s}\,ds \\ N &= k\ln s + C. \end{aligned}$$

Using our two initial conditions,

$$21 = k\ln 2.2 + C \tag{2}$$
$$13 = k\ln 8 + C. \tag{3}$$

This isn't as convenient as we have had in the past, but this is still a system of equations we know how to solve. Solving (2) for C and substituting this in (3), we have

$$\begin{aligned} 13 &= k\ln 8 + \underbrace{21 - k\ln 2.2}_{C} \\ k\ln 2.2 - k\ln 8 &= 21 - 13 \\ k(\ln 2.2 - \ln 8) &= 8 \\ k(\ln(2.2 - \ln 8)) &= 8 \\ k &= \frac{8}{\ln 2.2 - \ln 8} \\ k &\approx -6.196. \end{aligned}$$

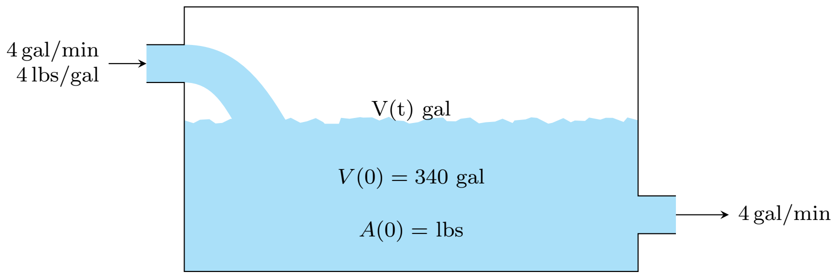
Plugging this back into (2) to find C,

$$\begin{aligned} C &= 21 - (-6.196)\ln 2.2 \\ &= 21 + 6.196\ln 2.2 \\ &= 25.8859. \end{aligned}$$

Now computing $N(11) = -6.196\ln 11 + 25.8859 = \boxed{11.03.}$ □

Example 5. A 500-gallon tank initially contains 340 gallons of pure distilled water. Brine containing 4 pounds of salt per gallon flows into the tank at the rate of 4 gallons per minute, and the well-stirred mixture flows out of the tank at the rate of 4 gallons per minute. Find the amount of salt in the tank after 5 minutes.

Solution. We again let $A(t)$ be the amount of salt in pounds at time t. Then since we have stuff flowing in at the same rate that there is stuff flowing out of the tank, the volume holds constant. Again we draw a picture.



Now

$$\begin{aligned}\frac{dA}{dt} &= \frac{4 \text{ lbs}}{\text{gal}} \cdot \frac{4 \text{ gal}}{\text{min}} - \frac{A(t) \text{ lbs}}{340 \text{ gal}} \cdot \frac{4 \text{ gal}}{\text{min}} \\ &= \left[16 - \frac{A(t)}{85}\right] \text{ gal/min.}\end{aligned}$$

We further know that $A(0) = 0$ since the tank is initially filled with distilled water. With this fact we have

$$\begin{aligned}\frac{dA}{dt} &= 16 - \frac{A}{85} \\ \int \frac{dA}{16 - \frac{A}{85}} &= \int dt \\ -85 \ln \left|16 - \frac{A}{85}\right| &= t + C \\ -85 \ln 16 &= C.\end{aligned}$$

In the last line we just plugged in 0 for both A and t. Now solving for A,

$$\begin{aligned}
-85\ln\left|16-\frac{A}{85}\right| &= t-85\ln 16\\
\ln\left|16-\frac{A}{85}\right| &= \frac{-t}{85}+\ln 16\\
16-\frac{A}{85} &= e^{-t/85+\ln 16}\\
16-\frac{A}{85} &= 16e^{-t/85}\\
\frac{A}{85} &= 16-16e^{-t/85}\\
A &= 1360-1360e^{-t/85}\\
&\boxed{A(5)\approx 77.69.}
\end{aligned}$$

□