Overview

In this lesson we move on from separable equations to another type of differential equation, namely *first-order linear differential equations.* We already know what a differential equation is. First-order just means that only the first derivative appears (so no y'', y''', etc). Linear means that y' and y are not multiplied together in any combination. For example $y' + ty = t^2 + 6$ is linear, but $yy' + y = 1$ and $y' + y^2 = 3t$ is not linear. So how do we solve such equations?

Lesson

If we are given a first-order linear equation, we can get it in the following form

$$y' + p(t)y = q(t). \tag{1}$$

Why do we want it in this form? Well, if we let $\mu(t) = e^{\int p(t)\,dt}$ then and multiply both sides of (1) by $\mu(t)$, we get

$$y'e^{\int p(t)\,dt} + e^{\int p(t)\,dt}p(t)y = e^{\int p(t)\,dt}q(t). \tag{2}$$

But the left hand side of (2) is precisely what we get if we computed $\frac{d}{dt}\left(ye^{\int p(t)\,dt}\right)$ using the product rule. So then we can rewrite (2) as

$$\begin{aligned}\left(ye^{\int p(t)\,dt}\right)' &= e^{\int p(t)\,dt}q(t)\\ (y\mu(t))' &= \mu(t)q(t),\end{aligned}$$

where in the second line we just used our definition of $\mu(t)$. Now integrating both sides, by the fundamental theorem of calculus, on the left hand side we'll just get $y\mu(t)$:

$$\begin{aligned}\int (y\mu(t))'\,dt &= \int \mu(t)q(t)\,dt\\ y\mu(t) &= \int \mu(t)q(t)\,dt + C. \end{aligned} \tag{3}$$

Definition. The term $\mu(t)$ is called an *integrating factor.*

To summarize:

Given an equation of the form

$$y' + p(t)y = q(t),$$

a solution is given by

$$y\mu(t) = \int q(t)\mu(t)\,dt,$$

where $\mu(t) = e^{\int p(t)\,dt}$.

How to solve first order linear equations

So the procedure goes as follows. In the wild we may come across a differential equation that looks like

$$a(t)y' + b(t)y = c(t).$$

Then we

1. Divide everything by $a(t)$ provided that $a(t) \neq 0$. This gives

$$y' + \frac{b(t)}{a(t)}y = \frac{c(t)}{a(t)},$$

   which is in the same form as (1).

2. Find the integrating factor by computing $\mu(t) = e^{\int p(t)\,dt}$, where $p(t) = \frac{b(t)}{a(t)}$.
3. Plug in the $\mu(t)$ you found into (3), where $q(t) = \frac{c(t)}{a(t)}$.
4. Integrate!
5. Divide both sides of the equation you have by $\mu(t)$.

> **Remark.** In this discussion we have used t's for the independent variable, but by this point we should be comfortable swapping t out for x or any other letter we want. Just be sure whatever the variable is in the problem that you stick with that variable.

Now let's see this in action with a few examples.

Example 1. Find a general solution to the differential equation

$$\frac{dy}{dx} + \frac{5}{x} = -2x + 5.$$

Solution. Here our $p(x) = \frac{5}{x}$ and $q(x) = -2x + 5$. So, assuming $x > 0$,

$$\mu(x) = e^{\int \frac{5}{x}\,dx} = e^{5\ln x} = e^{\ln x^5} = x^5.$$

Then we find a general solution by

$$\begin{aligned} yx^5 &= \int (-2x+5)x^5\,dx \\ yx^5 &= \int \left(-2x^6 + 5x^5\right)\,dx \\ yx^5 &= -\frac{2}{7}x^7 + \frac{5}{6}x^6 + C \\ y &= -\frac{2}{7}x^2 + \frac{5}{6}x + \frac{C}{x^5}. \end{aligned}$$

□

> **Remark.** In the above example we assumed that $x > 0$. Why is this an okay assumption? Well, we know we can't have $x = 0$ since $\frac{5}{x}$ appears in the differential equation. As you could check by using $\mu(x) = -x^5$, the only thing that would change in our final answer is we would get $-C/x^5$. But since C is just an arbitrary constant, we can just relabel $-C$ as C.

Example 2. Find the particular solution to the following initial value problem.

$$t^2y' + ty = 6, \qquad y(1) = 4$$

Solution. Here $p(t) = \frac{1}{t}$, and here we know $t > 0$ since $y(1) = 4$. So

$$\mu(t) = e^{\int \frac{1}{t}\,dt} = e^{\ln t} = t.$$

So

$$\begin{aligned} yt &= \int t \cdot \frac{6}{t^2}\,dt \\ yt &= \int \frac{6}{t}\,dt \\ yt &= 6\ln|t| + C \\ y &= \frac{6}{t}\ln|t| + \frac{C}{t}. \end{aligned}$$

Using $y(1) = 4$, we find that $C = 4$. So the general solution is

$$y = \frac{6}{t}\ln|t| + \frac{4}{t}.$$

□

Example 3. Find a general solution to the differential equation

$$y' - y = 19.$$

Solution. Here $p(x) = -1$, so $\mu(t) = e^{-x}$. And the general solution is given by

$$\begin{aligned} ye^{-x} &= \int 19e^{-x}\,dx \\ ye^{-x} &= -19e^{-x} + C \\ y &= -19 + Ce^{x}. \end{aligned}$$

□

> **Remark.** The previous example is also a separable equation, so we could have solved it using separation of variables as well.

Example 4. For $\frac{-\pi}{2} < x < 0$, find a general solution to the differential equation

$$y' + y\cot x = 7\csc x. \qquad (*)$$

Solution. Here $p(x) = \cot x$. So
$$\mu(x) = e^{\int \cot x\, dx}.$$
How do we compute $\int \cot x\, dx$? Recall that this is a simple u-substitution once we rewrite $\cot x = \frac{\cos x}{\sin x}$. We let $u = \sin x$ so $du = \cos x\, dx$. So
$$\int \cot x\, dx = \int \ln|\sin x|.$$
Combining this with $(*)$, we have
$$\mu(x) = \ln|\sin x|.$$
But we don't really like absolute values; can we get rid of them? Yes! on the interval $-\pi < x < 0$, $\sin x < 0$. So on this interval $|\sin x| = -\sin x$. So $\mu(x) = -\sin x$. From here it should be more straightforward.
$$\begin{aligned}
-y\sin x &= \int -7\sin x \csc x\, dx \\
-y\sin x &= \int -7\, dx && \sin x \csc x = 1 \\
-y\sin x &= -7x + C \\
y &= 7x\csc x + C\csc x.
\end{aligned}$$
Notice that we can keep the "$+C$" since we don't care what C is, so we can replace C by $-C$. □