Note. There should be no new information in this lesson. This is a brief review of things you should have learned in Calculus I, but certainly not exhaustive.

Derivatives and antiderivatives

There are several derivative anti derivative rules that you should have pretty well-memorized at this point:

Basic Differentiation Rules	Basic Integration Rules
$\frac{d}{dx}C = 0$	$\int 0\ dx = C$
$\frac{d}{dx}(kx) = k$	$\int k\ dx = kx + C$
$\frac{d}{dx}[kf(x)] = kf'(x)$	$\int [kf(x)]\ dx = k\int f(x)\ dx$
$\frac{d}{dx}[f(x) \pm g(x)] = f'(x) \pm g'(x)$	$\int [f(x) \pm g(x)]\ dx = \int f(x)\ dx \pm \int g(x)\ dx$
$\frac{d}{dx}x^n = nx^{n-1}$	$\int x^n\ dx = \frac{x^{n+1}}{n+1} + C, n \neq -1$ (Power Rule)
$\frac{d}{dx}\sin x = \cos x$	$\int \cos x\ dx = \sin x + C$
$\frac{d}{dx}\cos x = -\sin x$	$\int \sin x\ dx = -\cos x + C$
$\frac{d}{dx}\tan x = \sec^2 x$	$\int \sec^2 x\ dx = \tan x + C$
$\frac{d}{dx}\cot x = -\csc^2 x$	$\int \csc^2 x\ dx = -\cot x + C$
$\frac{d}{dx}\sec x = \sec x \tan x$	$\int \sec x \tan x\ dx = \sec x + C$
$\frac{d}{dx}\csc x = -\csc x \cot x$	$\int \csc x \cot x\ dx = -\csc x + C$
$\frac{d}{dx}e^x = e^x$	$\int e^x\ dx = e^x + C$
$\frac{d}{dx}\ln x = \frac{1}{x}, x > 0$	$\int \frac{1}{x}\ dx = \ln\|x\| + C$

It is very important that you know these well to make the transition into this course go smoothly.

Definition. Given a continuous function $f(x)$, if $F'(x) = f(x)$, we say that $F(x)$ is an *antiderivative* of $f(x)$, and we write

$$F(x) = \int f(x)\, dx.$$

Remark. If $F(x)$ is an antiderivative of $f(x)$ then the *indefinite integral* of f is given by

$$\int f(x)\,dx = F(x) + C,$$

where C is some constant. This means that any two functions which differ by only a constant will have the same antiderivative.

Remark. We want to note a couple things about the notation $\int f(x)dx$.

1. The process of computing $\int f(x)dx$ can be called
   - antidifferentiation
   - integration
   - taking or evaluating the integral
2. $f(x)$ is called the integrand.
3. Writing the "dx" on the outside is essential. This tells us that we are integrating with respect to x. It will especially be important in MA 16020 when we start having integrals with more than one variable.

Fundamental theorem of calculus

Theorem. If f is a real-valued continuous function on the interval $[a, b]$, and F the antiderivative of f:

$$F(x) = \int_a^x f(t)\,dt,$$

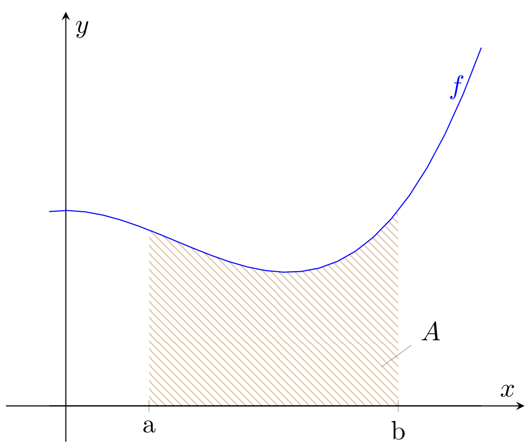
Then

$$F'(x) = f(x)$$

for all x in the interval (a, b). In particular, we have that

$$\int_a^b f(t)\,dt = F(b) - F(a).$$

This has an important geometric interpretation. Say we want to find the area under the curve f on the interval $[a, b]$ in the graph below. Then $A = \int_a^b f(x)\,dx$.



Example 1. Evaluate $\int_0^{\pi/3} \sec x(9\sec x + 6\tan x)\,dx$.

Solution. We start by distributing the $\sec x$, and then we should recognize some antiderivatives.

$$\begin{aligned}\int_0^{\pi/3} \sec x(9\sec x + 6\tan x)\,dx &= \int_0^{\pi/3} \left(9\sec^2 x + 6\sec x\tan x\right)\,dx \\ &= (9\tan x + 6\sec x)\Big|_0^{\pi/3} \\ &= 6 + 9\sqrt{3}\end{aligned}$$

□

Word problems

Everyone's favorite part of math is undoubtedly the word problems. The ones in this lesson test your understanding of what the integral represents. In such application problems we should know what it means if we take an integral. As a general rule, if you are given a rate, and you integrate, you get some sort of displacement.

Example 2. A faucet is turned on at 9:00 am and water starts to flow into a tank at a rate of

$$r(t) = 8\sqrt{t},$$

where t is time in hours after 9:00 am, and the rate $r(t)$ is in cubic feet per hour.

1. How much water, in cubic feet, flows into the tank from 10:00 am to 1:00 pm?
2. How many hours after 9:00 am will there be 92 cubic feet of water in the tank?

Solution. 1. Here we are given the rate $r(t)$ at which water flows into the tank. So if we compute $\int_a^b r(t)\,dt$, then we will find the total water added to the tank during the time interval $[a, b]$. Since time is given in hours after 9 am, here $a = 1$ and $b = 4$. Then

$$\int_1^4 8\sqrt{t}\,dt = 8\left(\frac{2}{3}\right)t^{3/2}\Big|_1^4 = 8\left(\frac{2}{3}\right)4^{3/2} - 8\left(\frac{2}{3}\right) = \frac{112}{3}.$$

2. Here we know the total amount of water in the tank, what we don't know is how long it took. So 92 is the value of our integral. Since we want the total amount in the tank, we should start at time $t = 0$. What we don't know is the end time. So

$$92 = \int_0^x 8\sqrt{t}\,dt. \qquad (*)$$

It may seem unnerving to have two variables in $(*)$, but it's okay because the t is going to go away when we integrate. Now

$$\begin{aligned}92 &= \int_0^x 8\sqrt{t}\,dt \\ &= 8\left(\frac{2}{3}\right)t^{3/2}\Big|_0^x \\ &= \frac{16}{3}x^{3/2}.\end{aligned}$$

Thus $\frac{276}{16} = x^{3/2}$, which means that

$$x = \left(\frac{276}{3}\right)^{2/3} \sim 6.676 \text{ hours}$$

□