SET OF DEFECT VALUES OF AN ENTIRE FUNCTION OF FINITE ORDER

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It is known that the set of defect values of a function meromorphic in the finite plane $\mathbb{C}$ is at most countable [1-3]. Arakelyan obtained the following result: for every countable set $A \subset \mathbb{C}$ and every $\rho > 1/2$ there exists an entire function of order ρ whose set of defect values contains A [4-6]. This theorem of Arakelyan disproved a conjecture of R. Nevanlinna. One is naturally led to asking whether there is an entire function of finite order whose set of defect values coincides with an arbitrarily prescribed countable set.

THEOREM. Let $A \subset \mathbb{C}$ be an at most countable set and let $\rho > 1/2$. Then there exists an entire function f of order ρ such that $\delta(a, f) > 0$ if and only if $a \in A$ or $a = \infty$.

The constraint $\rho > 1/2$ is essential since entire functions of order $\rho \leq 1/2$ cannot have finite defect values [3].

Proof. We assume that $\rho < \infty$ and the set A is infinite. For $\rho = \infty$ the theorem follows from a result of W. J. F. Fuchs and Hayman [2]. For a finite set A the result is well known.

Choose a number $\mu < \pi(1-1/(2\rho))$. Set $D_0 = \{z : |z| < 1\} \cup \{z : |z| < 2^3,\ 0 < \arg z < \mu\} \cup \{z : |z| > 2^6,\ -\mu < \arg z < 0\} \cup \{z : z| > 2^9,\ 0 \leqslant \arg z < \mu\}$.

Now fix an arbitrary sequence $\mu = \theta_1 > \theta_1' > \theta_2 > \theta_2' > \ldots > \theta_n > \theta_n' > \ldots \to 0$ and consider the domains $D_k^+ = \{z : 2^4 < |z| < 2^8,\ \theta_k' < \arg z < \theta_k\}$ and $D_k^- = \{z : 2 < |z| < 2^5,\ -\theta_k < \arg z < -\theta_k'\}$, $k = 1, 2, 3, \ldots$. Set $D_k = D_k^+ \cup D_k^-$ and $D = \bigcup_{k=0}^{\infty} D_k$.

We construct a subharmonic function w of order ρ which is positive in $\mathbb{C} \setminus \bar{D}$, equal to zero in D_0, and negative in D_k. To this end we first consider the domain $V = \{z : |z| < 1\} \cup \{z : |\arg z| < \mu\}$. We map the domain $\mathbb{C} \setminus \bar{V}$ in conformal and univalent manner onto the right half-plane (so that $\infty \to \infty$) and we let v denote the real part of the mapping function. Then v is a positive harmonic function in $\mathbb{C} \setminus \bar{V}$, and vanishes on the boundary. It is readily seen that

$$B(r, v) = \max_{|z|=r} v(z) = v(-r) \sim \text{const} \cdot r^{\lambda}, \quad r \to \infty, \tag{1}$$

where $\lambda = \pi/(2(\pi - \mu)) < \rho$. Let w denote the solution of the Dirichlet problem for the domain $\{z : |z| < 2^9\} \setminus \bar{D}$ with boundary conditions $w(z) = v(z)$ if $|z| = 2^9$ and $w(z) = 0$ if $z \in \partial D$. We extend w by 0 to D_0 and set $w(z) = v(z)$ for $|z| > 2^9$. The resulting function w is positive and subharmonic in $\mathbb{C} \setminus \bar{D}$ and equals 0 on the boundary $\partial(\mathbb{C} \setminus \bar{D})$. It remains to define w in D_k. Set $\delta_k = (\theta_k - \theta_k')/5$, $E_k^+ = \{z : 2^{4,1} \leqslant |z| \leqslant 2^{7,9},\ \theta_k' + \delta_k \leqslant \arg z \leqslant \theta_k - \delta_k\}$, $E_k^- = \{z : 2^{1,1} \leqslant |z| \leqslant 2^{4,9},\ -\theta_k + \delta_k \leqslant \arg z \leqslant -\theta_k' - \delta_k\}$, $E_k = E_k^+ \cup E_k^-$, $k = 1, 2, 3, \ldots$. Let w_k be a function continuous in $\bar{D}_k$, equal to zero on ∂D_k, equal to $-x_k < 0$ on E_k', and harmonic in $D_k \setminus \bar{E}_k'$. Here E_k' is a neighborhood of the set E_k, with $\bar{E}_k' \subset D_k$. It is not hard to show that if the numbers x_k are sufficiently small and decrease fastly as $k \to \infty$, then w_k gives a subharmonic extension of the function w to the domain D_k. The extended function is subharmonic in the entire plane and enjoys the following properties:

$$w(z) > 0, \quad z \in \mathbb{C} \setminus \bar{D}, \tag{2}$$

$$w(z) = 0, \quad z \in \bar{D}_0, \tag{3}$$

$$w(z) < 0, \quad z \in D_k, \quad k = 1, 2, \ldots, \tag{4}$$

$$w(z) = -x_k, \quad z \in E_k, \quad k = 1, 2, \ldots, \tag{5}$$

$$B(r, w) \leqslant cr^{\lambda}, \quad c > 0, \quad \lambda < \rho. \tag{6}$$

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Properties (2)-(5) follow from construction, and (6) is a consequence of (1). The Riesz measure of the function w is supported on the set $X_0=\partial D\cup\{z:|z|=2^9\}\cup\left(\bigcup_{k=1}^{\infty}\partial E_k'\right)$.

Set $u(z)=\sum_{n=1}^{\infty}2^{6n\rho}w(2^{-6n}z)$. It follows from (3) that for $|z|\leqslant 2^{6m}$ all the terms of the series with n ≥ m vanish. Hence, u is subharmonic in $\mathbb{C}$. Let us estimate u from above. Let $2^{6m}\leqslant|z|<2^{6(m+1)}$. By (2) and (6)

$$u(z)\leqslant c\sum_{n=1}^{m}2^{6n\rho}\cdot 2^{-6n\lambda}|z|^{\lambda}\leqslant c\cdot 2^{6m(\rho-\lambda)}|z|^{\lambda}(1+2^{6(\lambda-\rho)}+2^{12(\lambda-\rho)}+\ldots)\leqslant c_1\cdot 2^{6m(\rho-\lambda)}|z|^{\lambda}\leqslant c_1|z|^{\rho}. \qquad (7)$$

In what follows it will be very important that u possesses the following property:

$$u(2^{6n}z)=2^{6n\rho}u(z),\quad |\arg z|\leqslant\mu;$$
$$u(2^{6n}z)\geqslant 2^{6n\rho}u(z),\quad z\in\mathbb{C}, \qquad (8)$$

which follows from (3), the definition of u, and the definition of the domain D. The Riesz measure of the function u is supported on the set $X=\bigcup_{n=1}^{\infty}2^{6n}X_0$.

By a theorem of Azarin [7], there exists an entire function g such that

$$\log|g(z)|=u(z)+o(r^{\rho}),\quad r\to\infty, \qquad (9)$$

outside a set of disks of radii r_k centered at points z_k which satisfy the condition

$$\sum_{\{k:|z_k|<r\}}r_k=o(r),\quad r\to\infty. \qquad (10)$$

We denote the union of this exceptional set of disks by Z. An analysis of the proof given in [7] shows that the function g can be selected so that the centers of the exceptional disks will belong to the set X (which supports the Riesz measure).

Let $E_{k,n}=2^{6n}E_k$, $D_{k,n}=2^{6n}D_k$, $E_{k,n}^{\pm}=2^{6n}E_k^{\pm}$, and $D_{k,n}^{\pm}=2^{6n}D_k^{\pm}$. For any ε > 0 we let B(ε) denote the ε-neighborhood of the set B. Choose numbers ε_k such that the sets $D_k(\varepsilon_k)$ are pairwise disjoint, and consider the closed Jordan curves $\Gamma_{k,n}^{*}=2^{6n}\partial(D_k(\varepsilon_k))$. For fixed k we have, thanks to (8) and (2), that $\min\{u(z):z\in\Gamma_{k,n}^{*}\}\geqslant c_k 2^{6n\rho}$, $c_k>0$. Hence, one can find numbers $\varepsilon_{k,n}$ satisfying $\varepsilon_{k,n}\to 0$ as $n\to\infty$, $0<\varepsilon_{k,n}<\varepsilon_k$, such that for the curves $\Gamma_{k,n}^{\pm}=2^{6n}\partial(D_k^{\pm}(\varepsilon_{k,n}))$

$$\min\{u(z):z\in\Gamma_{k,n}^{\pm}\}\geqslant 2^{2n\rho},\quad n>n_0(k), \qquad (11)$$

$$\Gamma_{k,n}^{\pm}\cap Z=\varnothing,\quad n>n_0(k). \qquad (12)$$

It follows from (11), (12), and (9) that

$$\min\{|g(z)|:z\in\Gamma_{k,n}^{\pm}\}>\exp 2^{n\rho}=R_n,\quad n>n_0(k). \qquad (13)$$

It is readily seen that $E_{k,n}\cap Z=\varnothing$ for $n>n_0(k)$, and so, by (5), (8), and (9),

$$\log|g(z)|\leqslant(-x_k/2)\cdot 2^{6n\rho},\quad z\in E_{k,n},\quad n>n_0(k). \qquad (14)$$

LEMMA. Let $a\in\mathbb{C}$, $|a|<R/4$. Then there exists a univalent quasiconformal map α of the disk $\{z:|z|<R\}$ onto itself such that $\alpha(z)=z$ for $|z|=R$ and $\alpha(z)=R^2(z+a)/(R^2+4\bar{a}z)$ for $|z|=R/2$. The characteristic of this map does not differ from 1 by more than $64|a|/R$ throughout the disk $\{z:|z|<R\}$.

Such a map can be constructed explicitly and its characteristic can be calculated (cf. [3, Chap. VII, Sec. 2]).

Let $A=\{a_k\}_{k=1}^{\infty}$ be the set given in the statement of the theorem. Using the lemma, we construct for sufficiently large $n>n_0(k)$ maps $\alpha_{k,n}$ of the disks $\{z:|z|<R_n\}$ onto themselves such that $\alpha_{k,n}(z)=z$ for $|z|=R_n$,

$$\alpha_{k,n}(z)=R_n^2(z+a_k)/(R_n^2+4\bar{a}_k z),\quad |z|<R_n/2, \qquad (15)$$

and the characteristic of $\alpha_{k,n}$ does not differ from 1 by more than $64|a_k|/R_n$. Consider the components $G_{k,n}^{\pm}$ of the sets $\{z:|g(z)|<R_n\}$, which contain the respective sets $E_{k,n}^{\pm}$. By (14),

these components are not empty. It follows from (13) that the domains $G^{\pm}_{k,n}$ lie inside the curves $\Gamma^{\pm}_{k,n}$. We assume that the numbers $n_0(k)$ are so large that for $n > n_0(k)$ the domains $G^{\pm}_{n,k}$ are pairwise disjoint, relations (13)-(15) hold, and

$$\sum_{k=1}^{\infty} \sum_{n=n_0(k)}^{\infty} |a_k|/R_n < \infty. \tag{16}$$

Consider the function

$$g_1(z) = \begin{cases} (\alpha_{k,n} \circ g)(z), & z \in G^{\pm}_{k,n}, \quad n > n_0(k), \\ g(z), & z \notin \bigcup_{k=1}^{\infty} \bigcup_{n=n_0}^{\infty} (G^{+}_{k,n} \cup G^{-}_{k,n}). \end{cases}$$

It is continuous, since $\alpha_{k,n}(z) = z$ for $|z| = R_n$, and $|g(z)| = R_n$ for $z \in \partial G^{\pm}_{k,n}$. The function g_1 is locally a quasiconformal map everywhere, except for points $z_n \to \infty$, in the neighborhoods of which $g_1(z) = \varphi_m((z-z_m)^{p_m})$, where φ_m is a quasiconformal map and $p_m \in \mathbb{N}$. It maps the plane $\mathbb{C}$ quasiconformally onto a simply connected Riemann surface $\mathcal{F}$ which does not cover the point ∞. This surface is of parabolic type, and there exists an entire function f which maps $\mathbb{C}$ onto $\mathcal{F}$. in a conformal and univalent manner. The composition

$$f^{-1} \circ g_1(z) = \Phi(z) \tag{17}$$

maps the plane quasiconformally onto itself, and in view of (16) its characteristic p(z) satisfies the condition

$$\iint_{\mathbb{C}} (p(z) - 1) \frac{dxdy}{|z|^2} < \infty, \quad z = x + iy.$$

Hence, by a theorem of O. Teichmüller and Belinskii [8],

$$\Phi(z) \sim az, \quad z \to \infty, \quad a \in \mathbb{C} \setminus \{0\}. \tag{18}$$

Let us show that the function f constructed above enjoys all the desired properties. By (7), (9), (17), and (19)

$$\log M(r, f) = O(r^{\rho}), \quad r \to \infty. \tag{19}$$

It follows from (14) and (15) that

$$\log |g_1(z) - a_k|^{-1} \geq \frac{x_k}{3} \cdot 2^{6n\rho}, \quad z \in E_{k,n}.$$

Consequently,

$$\log |g_1(z) - a_k|^{-1} \geq x_k \cdot 2^{-12\rho} |z|^{\rho}, \quad z \in E_{k,n}, \tag{20}$$

and so $|z| \leq 2^{8+6n}$ whenever $z \in E_{k,n}$.

Consider the sets

$$T^{+}_{k} = \{z : 2^{4,5} \leq |z| \leq 2^{7,5}, \ \theta'_k + 2\delta_k \leq \arg z \leq \theta_k - 2\delta_k\},$$
$$T^{-}_{k} = \{z : 2^{1,5} \leq |z| \leq 2^{4,5}, \ -\theta_k + 2\delta_k \leq \arg z \leq -\theta'_k - 2\delta_k\},$$

where θ_k and δ_k are the numbers intervening in the definition of E_k. Set $T_k = T^{+}_{k} \cup T^{-}_{k}$ and $T^{*}_{k} = \bigcup_{n=n_0(k)}^{\infty} T_{k,n}$. By (17), (18), and (20), $\log |f(z) - a_k|^{-1} \geq c_k |z|^{\rho}$ on the set T^{*}_{k}. Since T^{*}_{k} intersects every sufficiently large circle $\{z : |z| = r\}$ along arcs whose angular measure is not less than δ_k, we have $m(r, a_k, f) \geq (2\pi)^{-1} c_k \delta_k r^{\rho}$. Using (19) this implies that $\delta(a_k, f) > 0$, and the order of the function f equals ρ. Moreover,

$$T(2r, f) \leq C_0 T(r, f) \tag{21}$$

with some $C_0 > 0$.

Let us show that there are no other defect values. To this end we use the following theorem of A. Edrei and W. H. J. Fuchs (see, for example, [3]): for every subset U of the unit circle whose measure does not exceed ε and every $a \in \mathbb{C}$

$$\int \log |f(re^{i\theta}) - a|^{-1} d\theta \leqslant C(\varepsilon) T(2r, f) + O(1), \quad r \to \infty,$$

where $C(\varepsilon) \to 0$ as $\varepsilon \to 0$.

Suppose $\delta(a, f) > 0$ but $a \notin A$. Choose a number $\varepsilon > 0$ so small that $(2\pi)^{-1} C_0 C(\varepsilon) < \delta(a, f)/2$, where C_0 is the constant appearing in (21). Let U be a set of measure ε consisting of a finite number of open intervals and containing all the points $0, \pm\theta_k, \pm\theta_k', k \in \mathbb{N}$. The complement of U in $[0, 2\pi]$ consists of finitely many segments. On each of these the function $f(2^{5.5+6n}e^{i\theta})$ tends uniformly as $n \to \infty$ to either one of the numbers $a_k \in A$, or to ∞. Hence, if we denote $r_n = 2^{5.5+6n}$, then $m(r_n, a, f) = (2\pi)^{-1} \int_U \log^+ |f(r_n e^{i\theta}) - a|^{-1} d\theta + O(1) \leqslant (2\pi)^{-1} C_0 C(\varepsilon) T(r_n, f) < (\delta(a, f)/2) \times T(r_n, f)$; contradiction. The theorem is proved.

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A PERIODIC BOUNDARY-VALUE PROBLEM FOR A CLASS OF DIFFERENTIAL-OPERATOR EQUATIONS

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In [1-3] periodic boundary-value problems for autonomous systems of differential equations were studied by the numerical-analytic method of Samoilenko [4].

In this paper we consider the question of the applicability of this method to the investigation of a periodic boundary-value problem for a differential-operator equation with right-hand side that does not depend explicitly on the time

$$dx/dt = f(x, Ax), \tag{1}$$

$$x(0) = x(T), \tag{2}$$

where A is an operator specified on the space of continuous functions.

The similar question for a nonautonomous differential-operator equation, namely for an equation of the form (1) with right-hand side $f(t, x, Ax)$, that depends explicitly on the time t and is periodic in t with period T, was considered in [5]. There an algorithm for finding periodic solutions of such equations was justified and existence theorems for periodic solutions of them were given.

The boundary-value problem (1), (2) can have a periodic solution with a period not known in advance, since the right-hand side of the equation does not contain the time t explicitly. On the other hand, if $x(t)$ is a periodic solution of (1), and h is a constant, then the function $x(t + h)$ is also a periodic solution of this equation. Because of these peculiarities the periodic boundary-value problem (1), (2) requires separate study.

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