We note that instead of the inequality $|t_p(\omega)| \leqslant 2\sqrt{p}$, whose proof is very involved, one could use the weaker inequality $|t_p(\omega)| \leqslant p^{4/5} + p^{1/5}$, proved in [6] by comparatively elementary means.

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MEROMORPHIC SOLUTIONS OF FIRST-ORDER ALGEBRAIC DIFFERENTIAL EQUATIONS

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Let K be the field of germs of meromorphic functions at ∞, M be the field of functions which are meromorphic for $r < |z| < \infty$ (r depends on the function). By $K[t_1, \ldots, t_n]$, $K(t_1, \ldots, t_n)$, respectively, we denote the ring of polynomials and field of rational functions in $t_1, \ldots, t_n$ over K. Let $y \in M$ be a solution of the differential equation $F(y', y) = 0$, $F \in K[t_1, t_2]$. We study the order of the function y, i.e., the order of growth of the Nevanlinna characteristic $T(r, y)$ as $r \to \infty$. The finiteness of the order is proved in [1], and it is established in [2] that the order is a rational number. The history of the question is described in [2, 3]. We note that in [2], instead of K the field $\mathbf{C}(z)$ is considered, and instead of M, the field of functions, meromorphic in $\mathbf{C}$. All the results and their proofs from [2] remain valid for the fields K and M also.

THEOREM. Let the function $y \in M$ satisfy a first order differential equation with coefficients from K. Then the order of the function y is a number of the form k/2 or k/3, with k a nonnegative integer.

For equations of the special form $(y')^m = R(y)$, $R \in K(t)$ this result is found in [3]. Examples from [3] show that all numbers of the form indicated can occur.

Proof. Since functions from K have order zero, one can assume that $y \in M \setminus K$. We consider the field $\mathfrak{A} = K(y', y) \subset M$. This field is of transcendence degree 1 over K, and is closed with respect to differentiation. By Theorem 6 of [2], the field $\mathfrak{A}$ is Fuchsian, so its genus is equal to 0 or 1 (cf. [2, Sec. 5]). We note that the orders of all elements of $\mathfrak{A} \setminus K$ are identical [2].

Let the field $\mathfrak{A}$ be of genus 0 over K. By a theorem of Lang [4, Chap. II, Paragraph 3.3 c] $\mathfrak{A} = K(w)$ holds for some $w \in \mathfrak{A}$. We have $w' = R(w)$, $R \in K(t)$. From the Fuchs conditions [2, 5], it follows that R is a quadratic polynomial, i.e., w is a solution of a Riccati equation with coefficients which are meromorphic at infinity. It is known that the order of any transcendental solution of such an equation is a number of the form k/2, $k \in \mathbf{N}$ [6], which proves the theorem in case of genus 0.

Now let the field $\mathfrak{A}$ be of genus 1 over K. We set $\mathfrak{A}_1 = \overline{K}(y', y)$, where $\overline{K}$ is the algebraic closure of the field K, i.e., the field of germs of functions which are algebraic at infinity. It is known [5, 7] that a Fuchsian differential field of genus 1 over an algebraically closed field $\overline{K}$ has the form $\mathfrak{A}_1 = \overline{K}(w, x)$, where

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$$w^2 = (x - e_1)(x - e_2)(x - e_3) = x^3 + Ax + B, \tag{1}$$

$e_i \in C$ are pairwise distinct, $e_1 + e_2 + e_3 = 0$,

$$x' = \lambda w, \quad \lambda \in \overline{K}, \quad \lambda(z) \sim \text{const}\, z^{\alpha}, \quad z \to \infty, \quad \alpha \in \mathbf{Q}. \tag{2}$$

It follows from (1), (2) that $x(z) = \wp\left(\frac{1}{2}\int\limits_{z_0}^{z}\lambda(t)\,dt\right)$, so $\alpha \geqslant -1$, and the order of the function x is equal to $2(1 + \alpha)$ [2, Sec. 5]. We shall prove that the number α is a multiple of 1/4 or 1/6.

For any $v \in \mathfrak{A}_1$, we denote by σv the result of analytic continuation of v along a curve going once around ∞. It follows from the obvious equation $\sigma y = y$ that σ is an automorphism of the field $\mathfrak{A}_1$. We note that σ commutes with differentiation. The general form of automorphisms of fields of genus 1 determined by (1) is known. Let $\Delta = 4A^3/(4A^3 + 27B^2)$. Any automorphism of the field $\mathfrak{A}_1$ is given by [8]

$$x_1 = \mu\,\{(w - b)^2/(x - a)^2 - x - a\}, \tag{3}$$

$$w_1 = \nu\,\{(w - b)/(x - a)\,[-(w - b)^2/(x - a)^2 + x + a] - (xb - wa)/(x - a)\}, \tag{4}$$

$$a, b \in \overline{K}, \quad b^2 = (a - e_1)(a - e_2)(a - e_3), \tag{5}$$

where $\mu = 1$, $\nu^2 = 1$ if $\Delta \neq 0, 1$; $\mu^3 = 1, \nu^2 = 1$ if $\Delta = 0$, $\mu^2 = 1, \nu^4 = 1$ if $\Delta = 1$.

Now let $x_1 = \sigma x$, $w_1 =$ w. We write the conditions on a and b under which the automorphism (3), (4) commutes with differentiation. For this we apply σ to (2) and into the formula obtained $x_1' = (\sigma\lambda)w_1$ we substitute the expressions for x_1', w_1 found from (3), (4). First let $b \neq 0$. After transformations considering (1), (2), (5), we get

$$(a')^2 = (\mu^{-1}\nu\sigma\lambda - \lambda)^2 (a - e_1)(a - e_2)(a - e_3).$$

(Compare [7, Sec. 13], where this calculation is made in the Jacobi notation.) Since the function a is algebraic at ∞, and $a \neq e_i$, i = 1, 2, 3 (by (5) and the assumption that $b \neq 0$), we get

$$\mu^{-1}\nu\sigma\lambda(z) - \lambda(z) = o(z^{-1}), \quad z \to \infty. \tag{6}$$

If b = 0, analogous calculations give $\mu^{-1}\nu\sigma\lambda = \lambda$, so in any case (6) holds. The number $\mu^{-1}\nu$ is a root of one of degree 4 or 6. Hence it follows from (6) that the number α is a multiple of 1/4 or 1/6, which is what had to be proved.

Now we consider an equation whose coefficients have an essential singularity at ∞, namely $F(y', y) = 0$, $F \in M[t_1, t_2]$. The solution of this equation $y \in M$ is called admissible, if for any coefficient a of the polynomial F one has $T(r, a) = o(T(r, y))$. In contrast with many results on first order algebraic differential equations (cf., e.g., the survey in [2]), the theorem proved does not have an analog for admissible solutions. In fact, in [9] A. A. Gol'dberg constructed an entire function w of preassigned order $\rho > 0$ with the following properties. The function N(r), counting simple roots of the equation $w^2(z) = 1$ in the disc $\{z: |z| \leqslant r\}$ has order zero. All remaining roots of this equation have multiplicity 2.

We consider the meromorphic function $a = (w'^2/(w^2 - 1)$. The functions counting poles of this function, obviously has order zero. Moreover, by the lemma on the logarithmic derivative [10], for the Nevanlinna approximation function $m(r, a)$ one has

$$m(r, a) \leqslant m(r, w'/(w - 1)) + m(r, w'/(w + 1)) = O(\ln r).$$

Thus, the function w has order zero, and the entire function w of preassigned order $\rho > 0$ is an admissible solution of the differential equation $(w')^2 = a(w^2 - 1)$.

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CONSTRUCTION OF CANONICAL COORDINATES ON ORBITS OF THE COADJOINT REPRESENTATION OF GRADED LIE GROUPS

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In this note we construct canonical coordinates on the simplest Ad*-orbits of Z_+-graded Lie algebras. Using a theorem of P. van Mörbecke, completely integrable systems on certain orbits of Borel subalgebras of simple Lie algebras are produced.

1. Let $\mathfrak{g}=\sum_{k\geqslant 0}\mathfrak{g}_k$ be a Z_+-graded Lie algebra and $\mathfrak{g}^*=\sum_{k\geqslant 0}\mathfrak{g}^*_{-k}$ be the dual space with the dual grading $\mathfrak{g}^*_{-k}=\left(\sum_{i\neq k}\mathfrak{g}_i\right)^\perp$. Obviously, for $f\in\mathfrak{g}^*_{-k}$ the stationary subalgebra $\mathfrak{g}_f$ is graded, $\mathfrak{g}_f=\sum\mathfrak{g}_{f,i}$, $\mathfrak{g}_{f,i}=\mathfrak{g}_f\cap\mathfrak{g}_i$, and for even k the space $\mathfrak{g}_{k/2}$ is orthogonal to $\sum_{i\neq k/2}\mathfrak{g}_i$ with respect to the form $\langle f, [.,.]\rangle$.

THEOREM 1. $\mathfrak{g}=\sum_{i\geqslant 0}\mathfrak{g}_i$ is a Z_+-graded Lie algebra. Suppose for $f\in\mathfrak{g}^*_{-k}$, $k>0$ there exists a $\mathfrak{g}_{f,0}$-invariant maximal isotropic subspace $\mathfrak{g}'_{k/2}$ of the space $\mathfrak{g}_{k/2}$ with respect to the form $\langle f,[.,.]\rangle$ ($\mathfrak{g}_{k/2}=0$ for odd k). Then

$$\mathfrak{p}=\sum_{i<k/2}\mathfrak{g}_{f,i}\dotplus\mathfrak{g}'_{k/2}\dotplus\sum_{j>k/2}\mathfrak{g}_j$$

is a polarization satisfying Pukanskii's condition.

Remarks. 1. If all operators of the algebra $\mathfrak{g}_{f,0}|\mathfrak{g}_{k/2}$ are nilpotent, the hypotheses of the theorem hold.

2. If the algebra $\mathfrak{g}$ is completely solvable and the grading is compatible with the filtration by derived series, the polarization $\mathfrak{p}$ can be constructed with the help of the construction of Vergne [7, 8], described in [1, 2].

3. The hypotheses of the theorem hold for Borel subalgebras of semisimple Lie algebras. This immediately gives the following

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