GROWTH OF ENTIRE AND SUBHARMONIC FUNCTIONS ON ASYMPTOTIC CURVES

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Let f be a meromorphic, transcendental function in the finite plane having a finite number of a-points. The well-known theorem of Iversen [1, p. 224] asserts that in this case there exists a curve Γ going out to ∞ such that $f(z) \to a$ as $z \to \infty$, $z \in \Gamma$. Such a curve is called an asymptotic curve. We henceforth make use of the standard notation from the theory of meromorphic functions [1] without special clarifications. The following questions arise naturally in connection with Iversen's theorem.

1°. If f is a rapidly increasing function, is it possible to choose the curve Γ such that f tends rapidly to the number a?

2°. Does Iversen's theorem remain true if the equation $f(z) = a$ has an infinite set of roots, but the growth of $N(r, a, f)$ is substantially less than the growth of the Nevanlinna characteristic $T(r, f)$?

These questions were posed by Hayman in a lecture at Moscow State Univ. in 1960 [2]. They were later repeated in [3] (Problems 2.6, 2.8). In answer to question 1°, Chang [4] proved that if the lower order of an entire function is equal to $\lambda > 0$, then there exists an asymptotic curve Γ for which

$$\varliminf_{\substack{z\to\infty\\ z\in\Gamma}} \ln\ln|f(z)|/\ln|z| \geqslant \min(1/2, \lambda).$$

In this inequality the right side cannot be replaced by a larger constant even under the additional assumption that f is an entire function without zeros [5].

Regarding question 2°, many results of negative character are known [1, Chap. V, Sec. 2]. The strongest counterexample is due to Hayman [6]. In this paper Hayman constructed an example of a meromorphic function f of order zero for which $\delta(\infty, f) = 1$ and ∞ is not an asymptotic value. In this same work Hayman proved the following theorem. If for a meromorphic function

$$\varliminf_{r\to\infty}\left\{T(r, f) - \frac{1}{2} r^{1/2} \int_r^\infty \frac{N(t, a, f)}{t^{3/2}}\, dt\right\} = +\infty,$$

then a is an asymptotic value of the function f. Hayman's theorem asserts nothing regarding the rate at which the function f tends to the asymptotic value. It follows easily from the theorem that if λ is the lower order of the function f and the order $N(r, f)$ is strictly less than $\min(1/2, \lambda)$, then ∞ is an asymptotic value. The constant $\min(1/2, \lambda)$ in the results of Chang and Hayman is best possible for the class of all functions of lower order λ.

In the present paper problems 1° and 2° are considered in the class of functions of order ρ and lower order λ, $0 \le \lambda \le \rho \le \infty$. Since Iversen's theorem is valid for subharmonic functions [7, 8], we shall consider problems 1° and 2° for such functions. In the proofs of the theorems of the present work analytic functions are not used, so that the main result carries over to subharmonic functions in $\mathbf{R}^m$, $m \ge 2$. However, the case $m \ge 3$ contains certain special features, and a different paper is devoted to it [14].

Remark. All the results of the present work were obtained independently of [4, 6]. The present work was completed in the summer of 1977 and was reported at a seminar on function theory at Rostov University (supervisor Professor M. G. Khaplanov) and at the Lvov Interschool Seminar on Function Theory (supervisor Professor A. A. Gol'dberg) in September-October of 1977.

Suppose that a function u can be represented in the form

$$u(z) = u_1(z) - u_2(z), \tag{0.1}$$

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where u_1, u_2 are subharmonic functions on **C**. We denote by μ_1 and μ_2, respectively, the positive and negative parts in the Jordan decomposition of the signed measure Δu, $\Delta u = \mu_1 - \mu_2$. We assume that the supports of the measures μ_1, μ_2 do not contain the origin. We set

$$n(r, u_i) = \mu_i\{z : |z| \leqslant r\}, \quad N(r, u_i) = \int_0^r n(t, u_i)\frac{dt}{t},$$

$$M(r, u_i) = \max_{|z|=r} u_i(z), \quad i = 1, 2.$$

$$T(r, u) = N(r, u_2) + \frac{1}{2\pi}\int_0^{2\pi} u^+(re^{i\theta})\,d\theta.$$

If f is meromorphic on **C**, then $\ln|f|$ can be represented in the form (0.1), and $n(r, u_1)$ and $n(r, u_2)$ are equal, respectively, to the number of zeros and poles (counting multiplicity) of the function f in the disk $\{z : |z| \leq r\}$.

For a function $\Phi(r)$ tending to $+\infty$ as $r \to \infty$ we define the order ρ and the lower order λ by the formulas

$$\left.\begin{matrix}\rho[\Phi]\\ \lambda[\Phi]\end{matrix}\right\} = \overline{\lim_{r\to\infty}}\ln\Phi(r)/\ln r.$$

The order (lower order) of a function u of the form (0.1) is called the order (lower order) of its characteristic $T(r, u)$. If u is a subharmonic function, its order and lower order do not change if $M(r, u)$ in place of $T(r, u)$ is used to define them. This follows from the well-known relation

$$1/3M(r/2, u) \leqslant T(r, u) \leqslant M(r, u). \tag{0.2}$$

We define the function $A(\rho, \lambda)$ for $0 \leq \lambda \leq \rho \leq \infty$ as follows:

$$A(\rho, \lambda) = \begin{cases} \lambda \quad \text{for} \quad \lambda \leqslant 1/2, & (0.3)\\ \rho - \dfrac{2(\rho - \lambda)}{1 + 2\sqrt{\lambda(1-\lambda)}} \quad \text{for} \quad \dfrac{1}{2} < \lambda < 1,\ \rho < \dfrac{\lambda + \sqrt{\lambda(1-\lambda)}}{2\lambda - 1}, & (0.4)\\ \dfrac{\rho}{2\rho - 1} \quad \text{for} \quad \dfrac{1}{2} < \lambda < 1,\ \rho \geqslant \dfrac{\lambda + \sqrt{\lambda(1-\lambda)}}{2\lambda - 1}, & (0.5)\\ \dfrac{\rho}{2\rho - 1} \quad \text{for} \quad \lambda \geqslant 1. & (0.6)\end{cases}$$

It is shown below (Lemma 5) that for $0 \leq \lambda \leq \rho \leq \infty$

$$\min(1/2, \lambda) \leqslant A(\rho, \lambda) \leqslant \min(1, \lambda). \tag{0.7}$$

THEOREM 1. Let the function u be subharmonic on **C** and have order ρ and lower order λ, $0 < \lambda \leq \rho \leq \infty$. There exists an asymptotic curve Γ for which

$$\underline{\lim_{\substack{z\to\infty\\ z\in\Gamma}}}\ln u(z)/\ln|z| \geqslant A(\rho, \lambda). \tag{0.8}$$

It follows from (0.7) that in the case of finite order Theorem 1 is stronger than the result of Chang. For $\rho = \infty$ (0.8) goes over into Chang's theorem.

THEOREM 2. Suppose that the function u has form (0.1) and has order ρ and lower order λ, $0 < \lambda \leq \rho \leq \infty$. Suppose that $\rho[N] = \rho[N(r, u_2)] < A(\rho, \lambda)$. Then there exists an asymptotic curve for which (0.8) is satisfied.

Estimate (0.8) is best possible for $\lambda \leq 1/2$ and for $\lambda \geq 1$.

For the case where $u(z) = \ln|f(z)|$, f a meromorphic function, Theorem 2 answers questions 1° and 2° (it may be assumed with no loss of generality that $a = \infty$). In this case the conditions for the existence of an asymptotic curve given by Theorem 2 and by Hayman's theorem [6] are not comparable.

It is obvious that Theorem 1 is a special case of Theorem 2. We shall show that Theorem 2, in turn, follows from Theorem 1. Suppose that the function u has the form (0.1). By means of a theorem of Hadamard [7, p. 142], we construct a subharmonic function u_3 of order $\rho[N]$ with associated measure μ_2. The function $u^* = u + u_3$ has order ρ and lower order λ by (0.7). This function is subharmonic on **C**, and Theorem 1 can be applied to it. Thus, there exists an asymptotic curve Γ for which

$$\underline{\lim_{\substack{z\to\infty\\ z\in\Gamma}}}\ln u^*(z)/\ln|z| \geqslant A(\rho, \lambda).$$

Since the order $\rho[u_3] = \rho[N] < A(\rho, \lambda)$, (0.8) is satisfied for the function $u(z) = u^*(z) - u_3(z)$.

Some auxiliary results needed for the proof of Theorem 1 are presented in Sec. 1. Theorem 1 is proved in Sec. 2. Talpur's method is used to construct the asymptotic curve in the case of subharmonic functions. Simplifications arising in the case of entire functions are indicated in each case. In Sec. 3 a version of Theorem 1 is presented which encompasses the case of zero lower order. Moreover, examples indicating the accuracy of Theorems 1 and 2 for $\lambda \geq 1$ are analyzed in Sec. 3. The construction of these examples is based on the method of Kennedy [9].

1

We set $D(r) = \{z : |z| < r\}$, $\bar{D}(r) = \{z : |z| \leq r\}$, $S(r) = \{z : |z| = r\}$, $K(r_1, r_2) = \{z : r_1 \leq |z| \leq r_2\}$. By the derivative of a function of a real variable we henceforth mean the right derivative.

Suppose that a function u is subharmonic in $D(r_0)$, $0 < r_0 \leq \infty$. We fix a number r_1, $0 < r_1 < r_0$. A component of the set $E = \{z : u(z) \geq M(r_1, u)\}$ is called thick if it contains a point z for which $u(z) > M(r_1, u)$. We shall consider only thick components of the set E and call them simply components. Let R be such a component. We set

$$v(z) = \begin{cases} u(z), & z \in R; \\ M(r_1, u), & z \in D(r_0) \setminus R. \end{cases} \tag{1.1}$$

The function v is subharmonic in $D(r_0)$ [7, p. 172]. Suppose that for the point $z_1 \in R$ the following relations are satisfied:

$$\left.\frac{d}{d \ln r}\right|_{r_1} M(r, u) = \left.\frac{d}{d \ln r}\right|_{r_1} M(r, v); \qquad |z_1| = r_1. \tag{1.2}$$

This implies that

$$u(z_1) = M(r_1, u). \tag{1.3}$$

In this case we call the point z_1 a principal point of the function u on the circle $S(r_1)$.

LEMMA 1. For each r_1, $0 < r_1 < r_0$, there exists at least one principal point of the function u on the circle $S(r_1)$.

Proof. Let $0 < r_1 < r_2 < r_0$. The set $E \cap K(r_1, r_2)$ has a finite number of thick components. This follows from Hayman's theorem [7, p. 175]. We denote these components by $R_1, \ldots, R_n$. We consider the sequence (a_j), $j \in \mathbb{N}$, $j \geqslant (r_2 - r_1)^{-1}$, of points for which $u(a_j) = M(|a_j|, u)$, $|a_j| = r_1 + j^{-1}$. Obviously, each point a_j belongs to one of the components $R_1, \ldots, R_n$. It is therefore possible to select a convergent subsequence (a_{j_k}), $a_{j_k} \in R$, where R is one of the sets $R_1, \ldots, R_n$. We set $z_1 = \lim_{k\to\infty} a_{j_k}$ and define the function v(z) by the relation (1.1). It is obvious that $z_1 \in R$, $|z_1| = r_1$, and hence (1.3) is satisfied. We have $M(r_1 + j_k^{-1}, u) = M(r_1 + j_k^{-1}, v)$. Because of the convexity of the functions $M(r, u)$, $M(r, v)$ with respect to the logarithm, from this we obtain (1.2). Thus, z_1 is a principal point of the function u on the circle $S(r_1)$. The proof of the lemma is complete.

For a function $u \geq 0$ subharmonic in $\bar{D}(r_0)$ we set

$$m(r, u) = \int_0^{2\pi} u^2(re^{i\theta})\, d\theta, \quad r \leqslant r_0.$$

The function u^2 is subharmonic in $D(r_0)$ [7, p. 46]. This implies that $m(r, u)$ is convex relative to the logarithm, and hence

$$dm(r, u)/d \ln r \geqslant 0, \; r \leqslant r_0. \tag{1.4}$$

LEMMA 2. Suppose that in the ring $K = K(x, y)$, $0 < x < y < \infty$, $y > ex$, there are continua R and R' each of which connects the circle S(x) with the circle S(y), and $R \cap R' = \varnothing$. Suppose that v and v' are subharmonic functions in K, $v(z) \geq 0$, $v'(z) \geq 0$; $v(z) = 0$ for $z \in K \setminus R$; $v'(z) = 0$ for $z \in K \setminus R'$. Then there exist positive functions l_1, l_2, $l_1(t) + l_2(t) = 2\pi$ such that

$$\ln M(y, v) \geqslant \pi \int_x^{y/e} \frac{dt}{t l_1(t)} + \frac{1}{2} \ln \left(\left.\frac{dm(r, v)}{d \ln r}\right|_x \right) - \frac{1}{2} \ln 2\pi, \tag{1.5}$$

$$\ln M(y, v') \geqslant \pi \int_x^{y/e} \frac{dt}{t l_2(t)} + \frac{1}{2} \ln \left(\frac{dm(r, v')}{d \ln r} \Big|_x \right) - \frac{1}{2} \ln 2\pi. \tag{1.6}$$

This lemma follows easily from Carleman's inequality for subharmonic functions [10]. (See also [1, p. 230] for the case $v(z) = \ln^+ |f_1(z)|$, $v'(z) = \ln^+ |f_2(z)|$, where f_1 and f_2 are holomorphic in $K(x, y)$.)

LEMMA 3. Suppose that the function u is subharmonic in $D(r_0)$, $r_0 > 1$, $u(z) \le 0$ for $|z| \le 1$. Then for all r, $1 < r \le r_0$,

$$dM(r, u)/d\ln r \geqslant M(r, u)/\ln r.$$

Proof. By the theorem on finite increments

$$\frac{dM(r, u)}{d \ln r} \geqslant \frac{M(r, u) - M(1, u)}{\ln r} \geqslant \frac{M(r, u)}{\ln r}.$$

LEMMA 4. Let $l_1(t) > 0$, $l_2(t) > 0$ be two functions on the interval $(0, \infty)$, $l_1(t) + l_2(t) = 2\pi$, and let $X \subset (0, \infty)$ be an arbitrary measurable set. Suppose that

$$\pi \int_X \frac{dt}{t l_2(t)} \leqslant p \int_X \frac{dt}{t}, \quad p > \frac{1}{2}. \tag{1.7}$$

Then

$$\pi \int_X \frac{dt}{t l_1(t)} \geqslant \frac{p}{2p-1} \int_X \frac{dt}{t}.$$

Proof. By the Cauchy–Bunyakovskii inequality we have

$$\int_X \frac{dt}{t l_j(t)} \int_X \frac{l_j(t)}{t} dt \geqslant \left(\int_X \frac{dt}{t} \right)^2, \quad j = 1, 2. \tag{1.8}$$

From (1.7), (1.8) with $j = 2$ it follows that

$$\int_X \frac{l_2(t)}{t} dt \geqslant \frac{\pi}{p} \int_X \frac{dt}{t}.$$

Hence,

$$\int_X \frac{l_1(t)}{t} dt = \int_X (2\pi - l_2(t)) \frac{dt}{t} \leqslant \left(2\pi - \frac{\pi}{p} \right) \int_X \frac{dt}{t}.$$

Substituting this inequality into (1.8) with $j = 1$, we obtain the assertion of the lemma.

Suppose there are given numbers ρ and λ, $0 \le \lambda \le \rho \le \infty$. We consider the extremal problem

$$A = \min \left\{ \frac{p}{2p-1} \left(1 - \frac{1}{t} \right) + \frac{\lambda}{t} : 1 \leqslant t \leqslant \infty,\ \frac{1}{2} \leqslant p \leqslant \infty,\ \rho t \geqslant p(t-1) + \lambda \right\}. \tag{1.9}$$

LEMMA 5. A solution of the extremal problem (1.9) is given by the function $A(\rho, \lambda)$ defined by relations (0.3)-(0.6). This function satisfies inequalities (0.7).

Proof. We first prove that the function $A(\rho, \lambda)$ defined by relations (0.3)-(0.6) satisfies condition (0.7). It is obvious that (0.7) is satisfied for $\lambda \le 1/2$ and for $\lambda \ge 1$.

Let $1/2 < \lambda < 1$, $\rho \le (\lambda + \sqrt{\lambda(1-\lambda)})/(2\lambda - 1)$. It is easy to see that in this case $dA(\rho, \lambda)/d\rho \le 0$, and therefore it suffices to verify (0.7) for $\rho = \lambda$ and for $\rho = (\lambda + \sqrt{\lambda(1-\lambda)})/(2\lambda - 1)$.

Suppose now that $1/2 < \lambda < 1$, $\rho > (\lambda + \sqrt{\lambda(1-\lambda)})/(2\lambda - 1)$. In this case the left inequality of (0.7) is obvious, while it suffices to verify the right inequality for $\rho = (\lambda + \sqrt{\lambda(1-\lambda)})/(2\lambda - 1)$, since the function $A(\rho, \lambda)$ $\rho/(2\rho - 1)$ is decreasing.

We shall first prove the first assertion of the lemma. We set $x = 1/t$, $q = p/(2p-1)$, $A = \min\limits_{1/2 \leqslant p \leqslant \infty} A(p)$, $A(p) = \min\{q + (\lambda - q)x :\ 0 \leqslant x \leqslant 1,\ p + (\lambda - p)x \leqslant \rho\}$. It is easy to see that $A = \lambda$ for $\lambda \le 1/2$. Let $\lambda > 1/2$. We have

$$A(p) = \lambda \quad \text{for} \quad p \leqslant \rho,\ \lambda \leqslant q; \tag{1.10}$$

$$A(p) = q \quad \text{for} \quad p \leqslant \rho,\ \lambda \geqslant q, \tag{1.11}$$

$$A(p) = \rho - 2(\rho - \lambda)\frac{p(1-p)}{(2p-1)(p-\lambda)} = \rho - 2(\rho - \lambda) B(p) \quad \text{for} \quad p \geqslant \rho. \tag{1.12}$$

It follows from (1.10) and (1.11) that

$$\min_{p \leqslant \rho} A(p) = \min(\lambda, \rho/(2\rho - 1)). \tag{1.13}$$

Investigating the function B(p) by elementary methods, we obtain

$$\max_{p \geqslant \rho} B(p) = B(\rho) \quad \text{for} \quad \lambda \geqslant 1; \tag{1.14}$$

$$\max_{p \geqslant \rho} B(p) = B(\rho) \quad \text{for} \quad 1/2 < \lambda < 1, \ \rho > (\lambda + \sqrt{\lambda(1-\lambda)})/(2\lambda - 1), \tag{1.15}$$

$$\max_{p \geqslant \rho} B(p) = B\left(\frac{\lambda + \sqrt{\lambda(1-\lambda)}}{2\lambda - 1}\right) = \frac{1}{1 + 2\sqrt{\lambda(1-\lambda)}}$$
$$\text{for} \quad 1/2 < \lambda < 1, \ \rho \leqslant \frac{\lambda + \sqrt{\lambda(1-\lambda)}}{2\lambda - 1}. \tag{1.16}$$

We note that

$$\rho - 2(\rho - \lambda) B(\rho) = \rho/(2\rho - 1). \tag{1.17}$$

From (1.12), (1.13), and (1.15) we obtain (0.4). From (1.12), (1.13), (1.15), and (1.17) we obtain (0.5). From (1.12), (1.13), (1.14), and (1.17) we obtain (0.6). The proof of the lemma is complete.

2

Proof of Theorem 1. We may assume with no loss of generality that

$$u(z) \leqslant 0 \quad \text{for} \quad |z| \leqslant 1. \tag{2.1}$$

We shall construct an asymptotic curve Γ whose existence is asserted in Theorem 1. We set $r_k = \exp k^2$, $k = 2, 3, \ldots$.

Let z_2 be a principal point of the function u on $S(r_2/4)$ which exists by Lemma 1. We shall construct by induction certain sequences of points (z_k) and curves (Γ_k), $\Gamma_k \subset \overline{D}(r_{k+1})$, such that the curve Γ_k joints the point z_k to the point z_{k+1}, $z_k \in S(r_k/4)$, and they possess a number of additional properties enumerated during the course of the construction.

Suppose that the points $z_2, \ldots, z_n$ and curves $\Gamma_2, \ldots, \Gamma_{n-1}$, $n \geq 2$, have already been constructed. We consider the closed set $E_n = \{z : u(z) \geq u(z_n)\}$. It is known [7, 11] that the thick components of the set E_n are unbounded. Let R_n be a component of the set $E_n \cap \overline{D}(r_{n+1})$ containing the point z_n. We set

$$v_n(z) = \begin{cases} u(z) - u(z_n), & z \in R_n; \\ 0, & z \in \overline{D}(r_{n+1}) \setminus R_n. \end{cases} \tag{2.2}$$

The function v_n is subharmonic in $D(r_{n+1})$. By Lemma 1 there exists a principal point $z_{n+1} \in S(r_{n+1}/4)$ of the function v_n.

We shall need the following result of Talpur [11; 7, p. 188].

THEOREM A. Suppose that u is subharmonic in a neighborhood N of the continuum R, and $u \geq M$ on R. Any two points of R can be joined by a curve on which $u \geq M - 1$.

Using this theorem, we draw a curve Γ_n joining the point z_n and the point z_{n+1} so that

$$u(z) \geqslant u(z_n) - 1, \ z \in \Gamma_n. \tag{2.3}$$

This completes the inductive construction.

Remark. In the case where $u(z) = \ln|f(z)|$ and f is an entire function, the set R_n is linearly connected, and it is therefore not necessary to use Theorem A. In place of (2.3) in this case it is possible to obtain the stronger relation $u(z) \geqslant u(z_n)$, $x \in \Gamma_n$.

We set $\Gamma = \bigcup_{k=2}^{\infty} \Gamma_k$ and prove that the curve Γ satisfies all the conditions of Theorem 1.

For each $k \geq 3$ we consider the component R_k^* of the set $R_k \cap \overline{D}(r_k)$ containing the point z_k. It is easy to see that

$$R_k^* \subset R_k, \quad R_k^* \subset R_{k-1}. \tag{2.4}$$

We define the function v_k^* as follows:

$$v_k^*(z) = \begin{cases} v_k(z), & z \in R_k^*; \\ 0, & z \in \overline{D}(r_k) \setminus R_k^*. \end{cases}$$

The functions v_k^* and $v_k - v_k^*$ are subharmonic in $\overline{D}(r_k)$. Since z_k is a principal point of the function v_{k-1}, it follows from (2.2) and (2.4) that $v_k^* = 0$ for $z \in D(|z_k|)$, and hence

$$M(r_k/4, v_k^*) = m(r_k/4, v_k^*) = 0. \tag{2.5}$$

Moreover, by (1.4) we have for $r_k/4 \le r \le r_k$

$$\frac{dm(r, v_k)}{d\ln r} = \frac{dm(r, v_k^*)}{d\ln r} + \frac{dm(r, v_k - v_k^*)}{d\ln r} \geqslant \frac{dm(r, v_k^*)}{d\ln r}. \tag{2.6}$$

Since for $r_k/4 \le r \le r_k$ we have $M(r, v_k^*) - M(r, v_{k-1}) = u(z_{k-1}) - u(z_k)$, it follows that

$$\frac{dM(r, v_k^*)}{d\ln r} = \frac{dM(r, v_{k-1})}{d\ln r}, \quad \frac{r_k}{4} \leqslant r \leqslant r_k. \tag{2.7}$$

For any subharmonic function we have by the Cauchy–Bunyakovskii inequality and (0.2)

$$m(r, w) \geqslant 2\pi T^2(r, w) \geqslant \frac{2\pi}{9} M^2(r/2, w). \tag{2.8}$$

We decompose the natural numbers $k \ge 2$ into two classes as follows.

1°. The number k we assign to the first class if the set R_k contains a principal point a_k of the function u on the circle $S(r_k/4)$. In this case a_k is a principal point not only of u but of v_k as well. Therefore,

$$M(r_k/4, u) = u(a_k) \leqslant \max\{u(z) : z \in R_k, \quad |z| = r_{k+1}/4\} = u(z_{k+1}), \tag{2.9}$$

$$\left.\frac{dM(r, u)}{d\ln r}\right|_{r_k/4} = \left.\frac{dM(r, v_k)}{d\ln r}\right|_{r_k/4}, \tag{2.10}$$

where the function v_k is defined in (2.2). By (2.1) the function u satisfies the hypotheses of Lemma 3. This lemma together with (2.10) gives

$$\left.\frac{dM(r, v_k)}{d\ln r}\right|_{r_k/4} \geqslant \frac{M(r_k/4, u)}{\ln(r_k/4)}. \tag{2.11}$$

2°. We assign a number k to the second class if the set R_k contains no principal point of the function u on the circle $S(r_k/4)$. Let a_k be a principal point of the function u on the circle $S(r_k/4)$. We denote by R_k' the component of the set $\{z : u(z) \geqslant M(r_k/4, u)\} \cap \overline{D}(r_{k+1})$ containing the point a_k. It is easy to see that $R_k \cap R_k' = \varnothing$ and that the continuum R_k' joins the circle $S(r_k/4)$ to the circle $S(r_{k+1})$. We set

$$v_k'(z) = \begin{cases} u(z) - M(r_k/4, u), & z \in R_k', \\ 0, & z \in \overline{D}(r_{k+1}) \setminus R_k'. \end{cases}$$

By means of Lemmas 1 and 3 we obtain the relation

$$\left.\frac{dM(r, v_k')}{d\ln r}\right|_{r_k/4} \geqslant \frac{M(r_k/4, u)}{\ln(r_k/4)}, \tag{2.12}$$

which is analogous to (2.11).

We can now prove (0.8). Because of (2.3) for this it suffices to show that

$$\ln u(z_{n+1}) \geqslant A(\rho, \lambda) \ln|z_{n+1}| + o(\ln|z_{n+1}|), \quad n \to \infty. \tag{2.13}$$

If the number n belongs to the first class, we have by (2.9), (0.7), and the choice of the numbers r_n

$$\ln u(z_{n+1}) \geqslant M(r_n/4, u) \geqslant \lambda \ln(r_n/4) + o(\ln|z_{n+1}|)$$

$$\geqslant A(\rho, \lambda) \ln(r_n/4) + o(\ln|z_{n+1}|) = A(\rho, \lambda) \ln|z_{n+1}| + o(\ln|z_{n+1}|), \quad n \to \infty.$$

Suppose now that the number n belongs to the second class. Let q be the largest number of the first class not exceeding n. Such a number q always exists, since the number 2 belongs to the first class by construction. Let k be a natural number, $q + 1 \le k \le n$. We apply Lemma 2 to the ring $K = K(r_k, r_{k+1}/4)$, setting there $R = R_k \cap K$, $R' = R_k' \cap K$, $v = v_k$, $v' = v_k'$. By this lemma there exists functions $l_{1k}(t)$, $l_{2k}(t)$, $l_{1k}(t) + l_{2k}(t) = 2\pi$

satisfying the inequalities (1.5) and (1.6). We set

$$l_1(t)=l_{1k}(t),\quad r_k\leqslant t\leqslant r_{k+1}/(4e),$$
$$l_2(t)=l_{2k}(t),\ r_k\leqslant t\leqslant r_{k+1}/(4e),\quad q+1\leqslant k\leqslant n.$$

From (1.6) it follows that

$$\ln M(r_{k+1}/4,u)\geqslant \ln M\left(r_{k+1}/4,v_k'\right)\geqslant \pi\int\limits_{r_k}^{r_{k+1}/(4e)}\frac{dt}{tl_2(t)}+\frac{1}{2}\ln\left(\frac{dm\left(r,v_k'\right)}{d\ln r}\bigg|_{r_k}\right)-\frac{1}{2}\ln 2\pi. \tag{2.14}$$

By the convexity of the function $m(r, v_k')$ relative to the logarithm

$$\frac{dm\left(r,v_k'\right)}{d\ln r}\bigg|_{r_k}\geqslant\frac{1}{\ln 4}\{m\left(r_k,v_k'\right)-m\left(r_k/4,v_k'\right)\}. \tag{2.15}$$

From the definition of the function v_k' it follows that $M(r_k/4, v_k') = m(r_k/4, v_k') = 0$. Using (2.15) and (2.8), we find that

$$\frac{dm\left(r,v_k'\right)}{d\ln r}\bigg|_{r_k'}\geqslant\frac{1}{\ln 4}m\left(r_k,v_k'\right)\geqslant\frac{2\pi}{9\ln 4}M^2\left(r_k/2,v_k'\right).$$

Again using the theorem on finite increments, the convexity of the function $M(r, v_k')$ relative to the logarithm, and inequality (2.12), we obtain

$$\frac{dm\left(r,v_k'\right)}{d\ln r}\bigg|_{r_k'}\geqslant\frac{\pi\ln 2}{9}\frac{M^2\left(r_k/4,u\right)}{\ln^2\left(r_k/4\right)}. \tag{2.16}$$

Substituting (2.16) into (2.14), we arrive at the inequality

$$\ln M(r_{k+1}/4,u)\geqslant\pi\int\limits_{r_k}^{r_{k+1}/(4e)}\frac{dt}{tl_2(t)}+\ln M(r_k/4,u)-\ln\ln r_k-Q, \tag{2.17}$$

where Q is an absolute constant. Summing formulas (2.17) on k for $q + 1 \leq k \leq n$, we find

$$\ln M(r_{n+1}/4,u)\geqslant\pi\sum_{k=q+1}^{n}\int\limits_{r_k}^{r_{k+1}/(4e)}\frac{dt}{tl_2(t)}+\ln M(r_{q+1}/4,u)-n\ln\ln r_n-nQ. \tag{2.18}$$

From (1.5) it follows that for each k, $q + 1 \leq k \leq n$, we have

$$\ln M(r_{k+1}/4,v_k)\geqslant\pi\int\limits_{r_k}^{r_{k+1}(4e)}\frac{dt}{tl_1(t)}+\frac{1}{2}\ln\left(\frac{dm\left(r,v_k\right)}{d\ln r}\bigg|_{r_k}\right)-\frac{1}{2}\ln 2\pi. \tag{2.19}$$

Using (2.6), the convexity of the function $m(r, v_k)$ relative to the logarithm, (2.5), (2.7), and (2.8), we obtain

$$\frac{dm\left(r,v_k\right)}{d\ln r}\bigg|_{r_k}\geqslant\frac{dm\left(r,v_k^*\right)}{d\ln r}\bigg|_{r_k}\geqslant\frac{1}{\ln 4}m\left(r_k,v_k^*\right)\geqslant\frac{2\pi}{9\ln 4}M^2\left(r_k/2,v_k^*\right)\geqslant\frac{\pi\ln 2}{9}\left(\frac{dM\left(r,v_k^*\right)}{d\ln r}\bigg|_{r_k/4}\right)^2=\frac{\pi\ln 2}{9}\left(\frac{dM\left(r,v_{k-1}\right)}{d\ln r}\bigg|_{r_k/4}\right)^2. \tag{2.20}$$

Applying Lemma 3 to the function v_{k-1} we obtain from (2.20)

$$\frac{dm\left(r,v_k\right)}{d\ln r}\bigg|_{r_k/4}\geqslant\frac{\pi\ln 2}{9}\frac{M^2\left(r_k/4,v_{k-1}\right)}{\ln^2\left(r_k/4\right)}.$$

We substitute the last inequality into (2.19). We have

$$\ln M(r_{k+1}/4,v_k)\geqslant\pi\int\limits_{r_k}^{r_{k+1}/(4e)}\frac{dt}{tl_1(t)}+\ln M(r_k/4,v_{k-1})-\ln\ln r_k-Q. \tag{2.21}$$

Summing (2.21) on k for $q + 2 \leq k \leq n$, we obtain

$$\ln M(r_{n+1}/4,v_n)\geqslant\pi\sum_{k=q+2}^{n}\int\limits_{r_k}^{r_{k+1}/(4e)}\frac{dt}{tl_1(t)}+\ln M(r_{q+2}/4,v_{q+1})-(n-1)\ln\ln r_n-(n-1)Q. \tag{2.22}$$

From (2.20) for k = q + 1 and (2.11) for k = q, using the fact that the number q belongs to the first class, we obtain

$$\frac{dm(r, v_{q+1})}{d\ln r}\bigg|_{r_{q+1}} \geqslant \frac{\pi\ln 2}{9}\left\{\frac{dM(r, v_q)}{d\ln r}\bigg|_{r_{q+1}/4}\right\}^2 \geqslant \frac{\pi\ln 2}{9}\left\{\frac{dM(r, v_q)}{d\ln r}\bigg|_{r_q/4}\right\}^2 \geqslant \frac{\pi\ln 2}{9}\{M^2(r_q/4, u)/\ln^2(r_q/4)\}. \tag{2.23}$$

From (2.23), (2.19) for k = q + 1, and (2.22), we obtain

$$\ln M(r_{n+1}/4, v_n) \geqslant \pi \sum_{k=q+1}^{n} \int_{r_k}^{r_{k+1}/(4e)} \frac{dt}{tl_1(t)} + \ln M(r_q/4, u) - n\ln\ln r_n - nQ. \tag{2.24}$$

We set $X_n = \bigcup_{k=q+1}^{n} [r_k, \quad r_{k+1}/(4e)]$. We note that

$$\int_{X_n} \frac{dt}{t} = \ln|z_{n+1}| - \ln|z_q| + o(\ln|z_{n+1}|),$$
$$n(\ln\ln r_n + Q) = o(\ln|z_{n+1}|), \quad n\to\infty. \tag{2.25}$$

We choose the number p_n from the condition

$$\pi\int_{X_n}\frac{dt}{tl_2(t)} = p_n\int_{X_n}\frac{dt}{t}.$$

Using Lemma 4, from this we obtain

$$\pi\int_{X_n}\frac{dt}{tl_1(t)} \geqslant \frac{p_n}{2p_n-1}\int_{X_n}\frac{dt}{t}, \quad p_n \geqslant \frac{1}{2}.$$

Inequalities (2.18) and (2.24) can now be rewritten in the form

$$\ln M(|z_{n+1}|, u) \geqslant p_n(\ln|z_{n+1}| - \ln|z_q|) + \ln M(|z_q|, u) + o(\ln|z_{n+1}|); \tag{2.26}$$

$$\ln M(|z_{n+1}|, v_n) \geqslant \frac{p_n}{2p_n-1}(\ln|z_{n+1}| - \ln|z_q|) + \ln M(|z_q|, u) + o(\ln|z_{n+1}|), \quad n\to\infty. \tag{2.27}$$

We observe that

$$\rho\ln|z_{n+1}| \geqslant \ln M(|z_{n+1}|, u) + o(\ln|z_{n+1}|), \ n\to\infty, \tag{2.28}$$
$$\lambda\ln|z_q| \leqslant \ln M(|z_q|, u) + o(\ln|z_{n+1}|), \ n\to\infty, \tag{2.29}$$
$$\ln u(z_{n+1}) \geqslant \ln M(|z_{n+1}|, v_n), \ n\to\infty. \tag{2.30}$$

We set $\ln u(z_{n+1}) = A_n \ln|z_{n+1}|$. From (2.26)-(2.30) it follows that for sufficiently large n

$$(\rho+\varepsilon_n)\ln|z_{n+1}| \geqslant p_n(\ln|z_{n+1}| - \ln|z_q|) + \lambda\ln|z_q|,$$
$$(A_n+\varepsilon_n)\ln|z_{n+1}| \geqslant \frac{p_n}{2p_n-1}(\ln|z_{n+1}| - \ln|z_q|) + \lambda\ln|z_q|,$$

where $\varepsilon_n \to 0$ as $n\to\infty$. We set $t_n = \ln|z_{n+1}|/\ln|z_q|$ and arrive at (1.9) with $\rho + \varepsilon_n$ in place of ρ, $A_n + \varepsilon_n$ in place of A, and p_n in place of p. It follows from Lemma 5 that $A_n + \varepsilon_n \geq A(\rho + \varepsilon_n, \lambda)$. Letting n tend to ∞, we obtain $\varliminf_{n\to\infty} A_n \geqslant A(\rho, \lambda)$, and hence $\ln u(z_{n+1}) \geqslant A(\rho, \lambda)\ln|z_{n+1}| + o(\ln|z_{n+1}|)$, $n\to\infty$, which proves (2.13) in the case where the number n is of second class. The proof of the theorem is complete.

3

By the method used in the proof of Theorem 1 it is possible to obtain the following result which encompasses the case of zero lower order.

THEOREM 3. Suppose that the positive, differentiable function φ is increasing on $[0, \infty)$,

$$\frac{d\varphi(r)}{d\ln r} \leqslant \frac{1}{2}, \tag{3.1}$$

and suppose for the subharmonic function u

$$\ln M(r, u) \geqslant \varphi(r), \quad 0 < r < \infty. \tag{3.2}$$

Then there exists an asymptotic curve Γ for which

$$\ln u(z) \geqslant \varphi(|z|)-(\ln|z|)^{1/2+\varepsilon}, \quad z \in \Gamma, \quad r > r(\varepsilon) \tag{3.3}$$

for any $\varepsilon > 0$.

Proof. We construct the curve Γ as in the proof of Theorem 1. Because of (2.3), it suffices to show that the relation (3.3) is satisfied at the points $z_k \in \Gamma$. If the number n belongs to the first class, we use (2.9), (3.1), and (3.2):

$$\ln u(z_{n+1}) \geqslant \ln M(|z_n|, u) \geqslant \varphi(|z_n|) \geqslant \varphi(|z_{n+1}|) - {}^1/_2(\ln|z_{n+1}| - \ln|z_n|)$$

$$= \varphi(|z_{n+1}|) - {}^1/_2((n+1)^2 - n^2) \geqslant \varphi(|z_{n+1}|) - n - {}^1/_2 \geqslant \varphi(|z_{n+1}|) - \ln|z_{n+1}|)^{1/2+\varepsilon}, \quad n \to \infty.$$

If the number n belongs to the second class, we have by (2.24), (2.25), and $l_1(t) \leq 2\pi$ the following inequalities:

$$\ln u(z_{n+1}) \geqslant \ln M(|z_{n+1}|, v_n) \geqslant {}^1/_2(\ln|z_{n+1}| - \ln|z_q|)\ \varphi(|z_{n+1}|) - \varphi(|z_q|) \leqslant {}^1/_2(\ln|z_{n+1}| - \ln|z_q|). \tag{3.4}$$

From (3.1) it follows that

$$\varphi(|z_{n+1}|) - \varphi(|z_q|) \leqslant 1/2(\ln|z_{n+1}| - \ln|z_q|).$$

Together with (3.4) this gives

$$\ln u(z_{n+1}) \geqslant \varphi(|z_{n+1}|) + O(n\ln n) \geqslant \varphi(|z_{n+1}|) - (\ln|z_{n+1}|)^{1/2+\varepsilon}, \quad n \to \infty.$$

The proof of the theorem is complete.

It is obvious that Theorems 1 and 2 are sharp for $\lambda \leq 1/2$. The following example shows that Theorem 1 is sharp for $\lambda \geq 1$. For such λ we have $A(\rho, \lambda) = \rho/(2\rho - 1)$.

Example. Let $\rho < \infty$. On $[0, \infty)$ we define two functions ψ_1 and ψ_2 as follows:

$$\psi_1(r) = \begin{cases} 1/\lambda, \ 8k \leqslant r \leqslant 8k+1; \\ (1/\rho - 1/\lambda)(r - 8k - 1) + 1/\lambda, \ 8k+1 \leqslant r \leqslant 8k+2; \\ 1/\rho, \ 8k+2 \leqslant r \leqslant 8k+3; \\ (1/\lambda - 1/\rho)(r - 8k - 3) + 1/\rho, \ 8k+3 \leqslant r \leqslant 8k+4; \\ 1/\lambda, \ 8k+4 \leqslant r \leqslant 8k+5; \\ (2 - 1/\rho - 1/\lambda)(r - 8k - 5) + 1/\lambda, \ 8k+5 \leqslant r \leqslant 8k+6; \\ 2 - 1/\rho, \ 8k+6 \leqslant r \leqslant 8k+7; \\ (1/\lambda - 2 + 1/\rho)(r - 8k - 7) + 2 - 1/\rho, \ 8k+7 \leqslant r \leqslant 8k+8. \end{cases}$$

$$k = 0, 1, \ldots;$$

$$\psi_2(r) = \begin{cases} 1/\lambda, \ 8k \leqslant r \leqslant 8k+1; \\ (2 - 1/\rho - 1/\lambda)(r - 8k - 1) + 1/\lambda, \ 8k+1 \leqslant r \leqslant 8k+2; \\ 2 - 1/\rho, \ 8k+2 \leqslant r \leqslant 8k+3; \\ (1/\lambda - 2 + 1/\rho)(r - 8k - 3) + 2 - 1/\rho, \ 8k+3 \leqslant r \leqslant 8k+4; \\ 1/\lambda, \ 8k+4 \leqslant r \leqslant 8k+5; \\ (1/\rho - 1/\lambda)(r - 8k - 5) + 1/\lambda, \ 8k+5 \leqslant r \leqslant 8k+6; \\ 1/\rho, \ 8k+6 \leqslant r \leqslant 8k+7; \\ (1/\lambda - 1/\rho)(r - 8k - 7) + 1/\rho, \ 8k+7 \leqslant r \leqslant 8k+8; \end{cases}$$

$$k = 0, 1, \ldots.$$

We set $l_j(r) = \psi_j(\ln^+\ln^+ r)$, $j = 1, 2$. It is obvious that

$$\left|\frac{dl_j(r)}{d\ln r}\right| \leqslant 1, \quad j = 1, 2. \tag{3.5}$$

We consider the half strips

$$\Omega_j = \left\{z = x + iy : |y| < \frac{\pi}{2} l_j(x), \quad x > e^e\right\}, \quad j = 1, 2.$$

Suppose that the function $\zeta_j(z)$ maps conformally and in single-sheeted fashion the half strip Ω_j onto the half strip $\Omega_0 = \{\zeta = \xi + i\eta : |\eta| < \pi/2, \ \xi > 0\}$, $\zeta_j(\pm i\pi/2\lambda + e^e) = \pm i\pi/2$, $\zeta_j(\infty) = \infty$. Applying a theorem of Ahlfors [12, p. 226] and a theorem of Varshavskii [12, p. 230], we obtain, on recalling (3.5),

$$\operatorname{Re}\zeta_j(x+iy) = (1+o(1))\int_0^x \frac{dt}{l_j(t)}, \quad j=1,2, \tag{3.6}$$

uniformly with respect to y as $x \to \infty$. We set $x_k = \exp\exp k$. Using the definition of the functions l_j, we deduce from (3.6):

$$\operatorname{Re}\zeta_j(x+iy) \leqslant (1+o(1))\rho x, \quad x\to\infty; \tag{3.7}$$
$$\max_j \{\operatorname{Re}\zeta_j(x+iy)\} \geqslant (1+o(1))\lambda x, \quad x\to\infty; \tag{3.8}$$
$$\operatorname{Re}\zeta_1(x_{8k+3}+iy) = (1+o(1))\rho x_{8k+3}, \quad k\to\infty; \tag{3.9}$$
$$\operatorname{Re}\zeta_j(x_{8k+1}+iy) = (1+o(1))\lambda x_{8k+1}, \quad k\to\infty; \tag{3.10}$$
$$\operatorname{Re}\zeta_1(x_{8k+7}+iy) = (1+o(1))(\rho/(2\rho-1))\,x_{8k+7}, \quad k\to\infty; \tag{3.11}$$
$$\operatorname{Re}\zeta_2(x_{8k+3}+iy) = (1+o(1))(\rho/(2\rho-1))\,x_{8k+3}, \quad k\to\infty. \tag{3.12}$$

The function $w = e^z$ maps the half strip Ω_1 onto a Jordan region D_1 in the **C** plane, $\infty \in \partial D_1$. The function $\zeta_1(\ln z)$ maps conformally and in single-sheeted fashion the domain D_1 onto the half strip Ω_0. We set $\sigma_1(r) = \max\{\operatorname{Re}\zeta_1(\ln z) : |z| = r\}$. By Kennedy's theorem [9, Theorem 2] there exists an entire function $F_1(z)$ bounded off the domain D_1 for which

$$\ln\ln M(r, F_1) \sim \sigma_1(r), \quad r\to\infty. \tag{3.13}$$

Similarly, the function $-e^z$ maps the half strip Ω_2 onto the domain D_2, $\infty \in \partial D_2$. It is easy to see that $D_1 \cap D_2 = \varnothing$. The function $\zeta_2(\ln(-z))$ maps conformally and in single-sheeted fashion the domain D_2 onto the half strip Ω_0. There exists an entire function $F_2(z)$ bounded off the domain D_2 for which

$$\ln\ln M(r, F_2) \sim \sigma_2(r), \quad r\to\infty, \tag{3.14}$$

where $\sigma_2(r) = \max\{\operatorname{Re}\zeta_2(\ln(-z)) : |z| = r\}$.

We set $F(z) = F_1(z) + F_2(z)$. From (3.7), (3.9), (3.13), and (3.14) it follows that the function F is of order ρ, and from (3.8), (3.10), (3.13), and (3.14) it follows that the lower order of the function F is equal to λ.

Let Γ be an asymptotic curve along which $F(z) \to \infty$. Since the function F is bounded on $\mathbf{C}\backslash(D_1 \cup D_2)$, starting from some point, the curve Γ is contained in one of the domains D_j. Suppose, e.g., that $\Gamma \subset D_1$. From (3.11), (3.13) it then follows that the order of growth of the function F on the curve Γ does not exceed $\rho/(2\rho - 1)$. If $\Gamma \subset D_2$, we use (3.12) and (3.14).

An analogous example can be constructed for $\rho = \infty$ as well by choosing suitable functions ψ_1 and ψ_2.

Thus, Theorem 1 is best possible for $\lambda \geq 1$. By applying to the example constructed above the method used in Example 2 of [1, Chap. V, Sec. 2], it is possible to construct an example demonstrating the sharpness of Theorem 2 for $\lambda \geq 1$. In this case for each $\varepsilon > 0$ we obtain a meromorphic function f of order ρ and lower order λ, $1 \leq \lambda \leq \rho \leq \infty$, for which $\rho[N] = \rho[N(r, f)] < \rho/(2\rho - 1) + \varepsilon$, and ∞ is not an asymptotic value of the function f.

In order to construct an analogous example with $\rho[N] = \rho/(2\rho - 1)$, it is necessary to invoke refined orders and in place of Lemma 2.1 of [1, Chap. V, Sec. 2] to use Theorem 5 of [13, Chap. II].

In the case where $1/2 < \lambda < 1$ it is not possible to construct corresponding examples. Probably Theorems 1 and 2 can be refined somewhat in this case.

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A REMARK ON RIGIDITY OF QUASICONFORMAL DEFORMATIONS OF DISCRETE ISOMETRY GROUPS OF HYPERBOLIC SPACES

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1. Mostow's well-known rigidity theorem [1] asserts that in more than two dimensions quasiconformal equivalence of two complete Riemannian manifolds of constant negative curvature and finite volume implies their isometry (up to homotopy). See [2] for strengthenings and generalizations of this theorem. In a recently published, very substantive preprint of Sullivan [3] is given, in particular, an extension of this theorem to the case of manifolds of infinite volume, whose volume grows slower than the volume of a hyperbolic space. Let us observe that the hypothesis about the validity of this result was first made in an equivalent form by the author in [4]. Sullivan's proof is based on very deep facts from ergodic theory. The indicated rigidity theorem admits an equivalent reformulation in terms of appropriate Kleinian groups after passage to universal coverings of the considered manifolds.

Here we show that in the more special case of manifolds, also, in general, of infinite volume, we can give an elementary proof of the rigidity theorem. This case covers Mostow's theorem, and for groups with a nonempty set of discontinuity on the invariant sphere somewhat sharpens the corresponding result of Sullivan. The proposed proof clears up the period of rigidity well.

2. Let $\mathcal{M}(n)$ be the Möbius group of (orientation-preserving) conformal automorphisms of the space $\overline{\mathbf{R}}^n = \mathbf{R}^n \cup \{\infty\}$, $n \geqslant 2$, and Γ be its discrete subgroup with the limit set $\Lambda(\Gamma)$ and the set of discontinuity $\Omega(\Gamma)$ (which can be empty).

The elements of $\mathcal{M}(n)$ are extended to isometries of the half space $\mathbf{R}_+^{n+1} = \{x = (x_1, \dots, x_{n+1}) : x_{n+1} > 0\}$, considered as a hyperbolic space with the metric

$$ds^2 = \sum_{j=1}^{n+1} dx_j^2 / x_{n+1}^2. \tag{1}$$

Then the extension of Γ gives a discontinuous group in $\mathbf{R}_+^{n+1}$; the boundary points of its fundamental polyhedron in $\mathbf{R}_+^{n+1}$, lying in $\Lambda(\Gamma)$, will be called its boundary vertices.

The group Γ is said to be (quasiconformally) rigid if each compatible (with it) quasiconformal automorphism $f : \mathbf{R}_+^{n+1} \to \mathbf{R}_+^{n+1}$, i.e., an automorphism f such that $f\Gamma f^{-1} = \Gamma_f \subset \mathcal{M}(n)$, induces a Möbius mapping on $\overline{\mathbf{R}}^n = \partial \mathbf{R}_+^{n+1}$ [i.e., for each such f the group Γ_f is conjugate to Γ in $\mathcal{M}(n)$].

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