THE INVERSE PROBLEM OF VALUE-DISTRIBUTION THEORY FOR MEROMORPHIC FUNCTIONS OF FINITE ORDER

A. É. Eremenko

UDC 517.535

In this article we give an almost complete solution of the inverse problem of the value distribution theory in the class of meromorphic functions of finite order. The problem that corresonds, roughly speaking, to the subclass of entire functions remains open.

1. We use the standard notation of the Nevanlinna theory (see, e.g., [1]). A meromorphic function, if not stated anything to the contrary, means a function that is meromorphic in the finite plane.

The following deficiency relation is one of the main results of the theory of meromorphic functions: For each meromorphic function f

$$\sum_{a\in\overline{\mathbf{C}}}\delta(a,f)\leqslant 2. \tag{1.1}$$

Moreover, $0\leqslant\delta(a,f)\leqslant 1$ for all $a\in\overline{\mathbf{C}}$.

Drasin [2] has solved the inverse problem of the value distribution theory, formulated in 1929 by Nevanlinna [3, p. 90]: To find, for each countable subset $\{a_j\}$ of $\overline{\mathbf{C}}$ and arbitrary numbers δ_j such that $0 < \delta_j \leqslant 1$ and $\sum_j\delta_j\leqslant 2$, a meromorphic function f, for which $\delta(a_j, f) = \delta_j$ and $\delta(a, f) = 0$ for $a\notin\{a_j\}$. The inverse problem was solved earlier for entire functions by Fuchs and Hayman [4]. The functions, constructed in [2, 4], have infinite order.

The solution of the inverse problem in the class of meromorphic functions of finite order is not less interesting. For example, let us observe that the deficiency of a function of infinite order is not a completely correct notion that characterizes the asymptotic behavior of the function, since it can strongly depend on the choice of the origin of coordinates. We can easily remove this dependence for meromorphic functions of finite order (see, e.g., [1, Chap. IV, Sec. 6]). Till now the inverse problem for meromorphic functions of finite order has been solved only in the case of a finite set of deficient values [1, Chap. VII, Sec. 5]. The difficulty of this problem in the case of an infinite set of deficient values is elucidated by the fact that the deficiencies of functions of finite order satisfy additional relations besides (1.1). Teichmüller [6] has even conjectured that for functions f of finite order

$$\sum_{a\in\overline{\mathbf{C}}}\delta^{\alpha}(a,f)<\infty \tag{1.2}$$

for $\alpha = 1/2$. The precise result in this direction has been obtained by Weitsman [7]: The relation (1.2) is valid with $\alpha = 1/3$ for meromorphic functions of finite lower order. The series in (1.2) can be divergent for $\alpha < 1/3$ [4]. Moreover, Weitsman [8] has proved that if equality is attained in (1.1) for a meromorphic function of finite lower order, then the set of deficient values is finite. Then Drasin [9] showed that in this case the deficiencies $\delta(a, f)$ must be rational numbers. On the other hand, Nevanlinna [10] has obtained the following result (see also [1, Chap. VII, Sec. 5]: Let there be given a finite set of complex numbers $a_1, \ldots, a_q$ and positive rational numbers $\delta_1, \ldots, \delta_q$ such that $\delta_j \leqslant 1$ and $\sum_{j=1}^{q}\delta_j=2$. Then there exists a meromorphic function f of finite order such that $\delta(a_j, f) = \delta_j$ for $1 \leqslant j \leqslant q$. In the present article, we prove the following theorem.

Kharkov. Translated from Sibirskii Matematicheskii Zhurnal, Vol. 27, No. 3, pp. 87-102, May-June, 1986. Original article submitted November 1, 1984.

THEOREM 1. Let $\{a_j\}_{j=1}^{\omega}$ be a countable subset of $\bar{\mathbf{C}}$ $(\omega \leqslant \infty)$ and δ_j be positive numbers such that

$$0<\delta_j<1,\ j=1,\ \ldots,\ \omega; \tag{1.3}$$

$$\Delta=\sum_{j=1}^{\omega}\delta_j<2; \tag{1.4}$$

$$\sum_{j=1}^{\omega}\delta_j^{1/3}<\infty. \tag{1.5}$$

Then there exists a meromorphic function f of finite order such that $\delta(a_j,\ f) = \delta$ and $\delta(a,\ f) =$ for $a \notin \{a_j\}$.

We give the proof of this theorem only for $\omega = \infty$. The proof in the case of finite ω is obtained by the same method (with simplifications). Moreover, Theorem 1 for $\omega < \infty$ follows from the mentioned result of Gol'dberg [5].

By virtue of (1.1), (1.2), and the Weitsman theorem [8], all the assumptions of Theorem 1 are necessary, except, possibly, the condition $\delta_j < 1$ in (1.3). Let us consider this condition in detail. If (1.4) is fulfilled, then the equality $\delta(a,\ f) = 1$ can be valid only for a single value of a. Let us suppose that $a_1 = \infty$ and $\delta_1 = 1$. The condition $\delta(\infty,\ f) = 1$ menas that f is similar to an entire function. We should obviously expect relations, stronger than (1.2), for these functions. Thus, Arakelyan [11] has put forward the conjecture that

$$\sum_{a\in\mathbf{C}}\frac{1}{\log\left(e/\delta(a,\ f)\right)}<\infty \tag{1.6}$$

for entire functions f of finite order. It is probable that the relation (1.6) is fulfilled for all meromorphic functions f of finite order such that $\delta(a,\ f) = 1$ for a certain $a \in \bar{\mathbf{C}}$.

The method of proof of Theorem 1 enables us to solve completely one more problem in the theory of meromorphic functions. Petrenko has studied the quantities

$$\beta(a,\ f)=\varliminf_{r\to\infty}\log^{+}M(r,\ a,\ f)/T(r,\ f),$$

where $M(r,\infty,f)=\sup_{|z|=r}|f(z)|$, $M(r,\ a,\ f)=M(r,\ \infty,\ (f-a)^{-1})$, and $a\in\mathbf{C}$. If f has finite lower order, then the set $E_n(f)=\{a\in\bar{\mathbf{C}}\colon \beta(a,\ f)>0\}$ is countable. The set E_n can have the cardinality of the continuum for functions f of infinite lower order. These results of Petrenko are given in [12]. Solving Petrenko's problem [12], the author [13] has proved that

$$\sum_{a\in\bar{\mathbf{C}}}\beta^{1/2}(a,\ f)<\infty$$

for meromorphic functions of finite lower order. We know [12] that the constant 1/2 in this relation cannot be replaced by a lesser one. The following theorem gives complete solution of the inverse problem for the quantities $\beta(a,\ f)$ in the class of meromorphic functions of finite order.

THEOREM 2. Let there be given a countable subset $\{a_j\}_{j=1}^{\omega}$ of $\bar{\mathbf{C}}$ $(\omega \leqslant \infty)$ and numbers $\beta_j > 0$ such that

$$\sum_{j=1}^{\omega}\beta_j^{1/2}<\infty.$$

Then there exists a meromorphic function f of finite order such that $\beta(a_j,\ f) = \beta_j$ and $\beta(a,\ f) = 0$ for $a\notin\{a_j\}$.

The proof of Theorems 1 and 2 is based on the application of the so-called pseudomeromorphic functions. This method, published in the articles of Pöschl and Wittich in the fourties, was applied for the first time to the inverse problem by Le Van Thiem [14]. By now the use of pseudomeromorphic functions has become a basic tool for the solution of the inverse problem of the value distribution theory [1, 2, 5, 14]. Necessary information on quasiconformal mappings is contained in [1, Chap. VII, Sec. 15]. We need only piecewise-smooth quasiconformal mappings.

A continuous function g in a domain $D \subset \mathbf{C}$ is said to be pseudomeromorphic if there exists a discrete subset X of D such that each point $X \subset D$ has a neighborhood V for which the restriction $z \in D \setminus X$ is a (univalent) quasiconformal mapping. If $D = \mathbf{C}$, then all these functions have the representation

$$g = f \circ \varphi, \tag{1.7}$$

where $\varphi: \mathbf{C} \to \mathbf{C}$ is a quasiconformal homeomorphism and f is a meromorphic function [16]. For each pseudomeromorphic function g, the characteristic $p_g(z) = (|g_z| + |g_{\bar{z}}|)/(|g_z| - |g_{\bar{z}}|)$ is defined almost everywhere. The Teichmüller–Belinskii theorem [15] states that if

$$\iint\limits_{|z|>r_0} (p_g(re^{i\theta}) - 1) \frac{dr d\theta}{r} < \infty \tag{1.8}$$

for a certain $r_0 > 0$, then there exists a representation (1.7), in which φ is a homeomorphism such that

$$\lim_{z\to\infty} \frac{\varphi(z)}{z} = 1. \tag{1.9}$$

The further treatment follows the following plan. In Sec. 2 we construct meromorphic functions that depend on certain parameters. In Secs. 3 and 4 we prove Theorems 1 and 2 respectively with the help of suitable choice of these parameters.

2. Let there be given sequences of positive numbers (x_j) and (θ_j), $j \in \mathbf{Z}$, a sequence (b_j), $j \in \mathbf{Z}$, of points of the extended complex plane, an even natural number M, and positive numbers x and ρ such that

$$x_1 = x_2 = \ldots = x_M = x; \tag{2.1}$$

$$\theta_1 = \theta_2 = \ldots = \theta_M = \pi/\rho; \quad \theta_j \leqslant \pi/\rho, \ j \in \mathbf{Z}; \tag{2.2}$$

$$\frac{x_1}{2} + \sum_{j=0}^{\infty} x_{-j} + \sum_{j=1}^{M/2} x_{2j} + \sum_{j=M+1}^{\infty} x_j = 1; \tag{2.3}$$

$$\sum_{j=-\infty}^{\infty} \theta_j = \pi; \tag{2.4}$$

$$b_1 = 0; \ b_M = \infty; \tag{2.5}$$

$$b_{j+1} \neq b_j, \ 1 \leqslant j \leqslant M-1; \tag{2.6}$$

$$|b_j| > 4, \ j \geqslant M+1; \ |b_j| < 5, \ j \leqslant 0. \tag{2.7}$$

The set $\{j \in \mathbf{Z}: b_j = a\}$ is finite for each $a \in \overline{\mathbf{C}}$.

Starting from these data, we construct a meromorphic function f of order ρ with the following properties:

$$T(r, f) = (2 + o(1))(\pi\rho)^{-1} r^{\rho}, \ r \to \infty; \tag{2.8}$$

$$m(r, a, f) = (2 + o(1))(\pi\rho)^{-1} r^{\rho} \sum_{\{k: b_k = a\}} x_k (1 - \cos(\rho\theta_k/2)), \ r \to \infty. \tag{2.9}$$

Let us set

$$\varphi_k = \sum_{j=-\infty}^{k} \theta_j, \ \psi_k = \frac{1}{2}(\varphi_{k-1} + \varphi_k), \ k \in \mathbf{Z}. \tag{2.10}$$

The third property of the function f, which we propose to construct, is the following one:

$$\log|f(re^{i\varphi}) - b_k|^{-1} = (x_k + o(1)) r^{\rho} \sin\left(\rho\left(\frac{\theta_k}{2} - ||\varphi| - \psi_k|\right)^+\right). \tag{2.11}$$

Here $r \to \infty$ and the relation (2.11) is fulfilled uniformly with respect to φ in arbitrary angles of the form

$$0 < \varepsilon < ||\varphi| - \psi_k| < \theta_k/2 - \varepsilon, \ k \in \mathbf{Z}, \ \varepsilon > 0.$$

If $b_k = \infty$, then the left-hand side of (2.11) should be replaced by $\log|f(re^{i\varphi})|$. In the sequel, we will not specifically mention about this modification in analogous formulas.

Let us set $D_k=\{z: \varphi_{k-1}<\arg z<\varphi_k\}$ and $D_k^*=\{z: -\varphi_k<\arg z<-\varphi_{k-1}\}$ for $k\in Z$. The closures of the angles D_k and D_k^* fill the whole plane, except the real axis. The bisector of the angle $D_k(D_k^*)$ is given by the equation $\arg z = \psi_k$ ($\arg z = -\psi_k$). Let E_k be sufficiently small (pairwise disjoint) angles with the bisectors $\{z: \arg z=\varphi_k\}$. Let $n(r, a, E, f)$ denote the number of the a-points of f in the set $E\cap\{z: |z|\leq r\}$, and $N(r, a, E, f)$ denote the corresponding Nevanlinna number function. The construction of f is carried out in several steps (Paragraphs 1-5). Everywhere in the sequel, taking liberty with the language, we will say that a function, defined in $\mathbb{D}$, satisfies the condition (1.8), meaning that the integration in (1.8) is taken over $\mathbb{D}$.

1. At first, the construction of the desired function is carried out in the angles

$$G_1=\{z: \psi_1<\arg z<\psi_M\},\ G_1^*=\{z:\bar{z}\in G_1\}.$$

LEMMA 1. There exists a pseudomeromorphic (in the domain G_1) function g_1 with the following properties: The characteristic p_{g_1} satisfies the condition (1.8);

$$N(r, a, E_k, g_1)\sim(2\pi\rho)^{-1}x_k r^\rho,\ r\to\infty,\quad a\notin\{b_j\},\ 1\leq k\leq M-1; \tag{2.12}$$

$$N\left(r, a, G_1\setminus\bigcup_{k=1}^{M-1}E_k, g_1\right)=O(\log r), r\to\infty; \tag{2.13}$$

$$\log|g_1(re^{i\varphi})-b_k|^{-1}\sim x_k r^\rho\sin(\rho(\varphi-\varphi_{k-1})) \tag{2.14}$$

for $r\to\infty$ uniformly with respect to φ inside the angles $|\varphi-\psi_k|\leq\theta_k/2-\varepsilon$, $\varepsilon>0$, $1\leq k\leq M$;

$$g_1(re^{i\psi_1})=\exp(-xr^\rho),\ r>r_0; \tag{2.15}$$

$$g_1(re^{i\psi_M})=\exp(xr^\rho),\ r>r_0. \tag{2.16}$$

Proof. We follow [1, Chap. VII, Sec. 5]. Let ε, $0<\varepsilon<1$, be so small that the closed disks C_k of radius ε with centers at finite points b_k, $1\leq k\leq M$, are pairwise nonintersecting and are all contained in the disk $\{z: |z|<\varepsilon^{-1}\}$. If $b_k=\infty$, then we set $C_k=\{z\in\bar{\mathbb{C}}: |z|\geq\varepsilon^{-1}\}$. In each disk C_k we draw the radius $\lambda_k=\{b_k+t: 0<t\leq\varepsilon\}$. If $b_k=\infty$, then $\lambda_k=[\varepsilon^{-1},\infty)$. Let us denote the point of intersection of the radius λ_k and the circle ∂C_k by b_k^*. We join the points b_k^* and b_{k+1}^* by a simple smooth curve Λ_k that does not intersect any of the disks C_j, $1\leq J\leq M$, so that the curve $\Lambda_k\cup\lambda_k\cup\lambda_{k+1}$ is smooth. The curve Λ_k is oriented from b_k^* to b_{k+1}^*. On the Riemann surface of the function $\log((w-b_k)/(w-b_{k+1}))$ ($\not\equiv$ const by virtue of (2.6)) we draw a cut that projects into the curve $\Lambda_k\cup\lambda_k\cup\lambda_{k+1}$. In addition, the Riemann surface splits into two parts – "the logarithmic ends." Let us denote the part that abuts on the curve Λ_k on the right by $\mathcal{F}_k$. Let us map the Riemann surface $\mathcal{F}_k$ quasiconformally onto the upper half plane. To this end, let us consider the subsets Ω_k', $\Omega_k''\subset\mathcal{F}_k$ that lie over the disks C_k and C_{k+1} respectively. The function $z=-\log(w-b_k)$ maps Ω'_k conformally onto the quadrant $\Pi'=\{x+iy: y>0,\ x>-\log\varepsilon\}$. If $b_k=\infty$, then it is necessary to use the function $z=\log w$. In the same way, the function $z=\log(w-b_{k+1})$ ($z=-\log w$, if $b_{k+1}=\infty$) maps Ω_k'' conformally onto the quadrant $\Pi''=\{x+iy: y>0,\ x<\log\varepsilon\}$. In order to extend the mapping to the remaining part of the surface $\mathcal{F}_k$, let us consider the curvilinear quadrilateral $Q_k=\bar{\mathbb{C}}\setminus(C_k\cup C_{k+1}\cup\Lambda_k)$. Two sides of this quadrilateral are the circles ∂C_k and ∂C_{k+1} and the other two sides are the edges of the curve Λ_k. Let us map the quadrilateral Q_k quasiconformally onto the rectangle $R_1=\{x+iy: 0<y<2\pi,\ |x|<-\log\varepsilon\}$ such that dilatation on the circles ∂C_k and ∂C_{k+1} is constant and these circles transform into the right and the left vertical sides respectively of the rectangle R_1. Moreover, we require that the dilatation is constant on the edges of the curve Λ_k. The left edge transforms into the upper horizontal side of the rectangle R_1 and the right edge transforms into the lower one. It is easily seen that under this mapping a pair of points that are pasted on ∂Q_k transform into points with the same abscissa. Let us denote the mapping function by α_k: $Q_k\to R_1$. The surface $\mathcal{F}_k\setminus(\Omega_k'\cup\Omega_k'')$ consists of a denumerable set of open quadrilaterals Q_k^j that project into Q_k and a denumerable set of curves that project into Λ_k. We suppose that the quadrilaterals Q_k^j are numbered such that Q_k^{j+1} and Q_k^j, $j\in\mathbb{N}$, have a part of boundary in common. Let us map the quadrilateral Q_k^j onto the rectangle $R_j=R_1+2\pi i(j-1)$ with the help of the function $\alpha_k+2\pi i(j-1)$. It is easily seen that the quasiconformal mapping $\mathcal{F}_k\to C^+=\{z: \operatorname{Im} z>0\}$ is constructed in this manner. Let H_k denote the inverse mapping. The pseudomeromorphic function H_k is holomorphic in $\Pi'\cap\Pi''$ and has bounded characteristic at the remaining points of the half plane. Since the set $C^+\setminus(\Pi'\cup\Pi'')$ has finite logarithmic area (i.e., the integral of the function $|z|^{-2}$ over this set is convergent), the function

H_k has the property (1.8). Simple computation shows that for each angle E whose bisector is the positive imaginary semiaxis and for arbitrary $a \neq b_k$, b_{k+1} we have

$$N(r, a, E, H_k) \sim (2\pi)^{-1} r, \; r \to \infty, \tag{2.17}$$

and only a finite set of a-points lies outside E.

Now let μ_k be that branch of the function $(z \exp(-i\psi_k))^\rho$ which maps the angle $\Delta_k = \{z: \psi_k < \arg z < \psi_{k+1}\}$ conformally onto $\dot{C}^+$. Let us define the function g_1 as follows:

$$g_1(z) = H_k(\mu_k(x_k^{1/\rho} z)), \; z \in \Delta_k, \; 1 \leqslant k \leqslant M-1.$$

It is obvious from the construction of the functions H_k that the function g_1 is continuous (and is even holomorphic) on the sides of the angles Δ_k for $|z| > r_0$. Changing the function g_1 on a bounded set, we can make it pseudoholomorphic in G_1. The function g_1 has the property (1.8) because all the functions H_k have this property. The relations (2.12) and (2.13) follow from (2.17), and (2.14)-(2.16) are verified directly. The lemma is proved.

Let us now construct the analogous pseudomeromorphic function g_1^* in the domain G_1^*. The function g_1^* has the properties (1.8) and (2.12)-(2.16) with E_k replaced by $E_k^* = \{z: \bar{z} \in E_k\}$; ψ_1 replaced by $-\psi_1$, and ψ_M replaced by $-\psi_M$.

2. Let us consider the function

$$H(re^{i\varphi}) = H(re^{i\varphi}, x, \theta) = xr^\rho \sin(\rho(\theta/2 - |\varphi|)^+),$$

where $|\varphi| \leqslant \pi$ and $0 < \theta \leqslant \pi/\rho$. This function can be expressed as a difference of two subharmonic functions. The positive part of the Riesz charge is concentrated on the rays $\arg z = \pm\theta/2$ and has density $(2\pi)^{-1}\rho x r^{\rho-1}$ on each of these rays. The negative part of the charge is concentrated on the positive ray and has density $\pi^{-1}\rho x(\cos(\rho\theta/2))r^{\rho-1}$ on this ray. We need a meromorphic function, the logarithm of whose modulus approximates H well.

<u>LEMMA 2.</u> Let $\mu(t) \uparrow \infty$, $t > 0$. Then the integral

$$u(z) = \int_1^\infty \log\left|1 - \frac{z}{t}\right| d(\mu(t) - [\mu(t)])$$

is convergent for $z \notin \mathbf{R}^+$ and satisfies the estimate $|u(z)| = O(\log|z|)$, $z \to \infty$, uniformly with respect to arg z in each domain of the form $\mathbf{C}\setminus\{z: \operatorname{Re} z > -a, |\operatorname{Im} z| < a\}$, $a > 0$.

<u>Proof</u>. Integrating by parts, we get

$$u(z) = -\operatorname{Re}\int_1^\infty \frac{z(\mu(t) - [\mu(t)])\,dt}{t(t-z)}.$$

Consequently,

$$|u(z)| \leqslant \int_1^\infty \frac{|z|\,dt}{t|t-z|} = O(\log|z|), \; z \to \infty,$$

in the domain under consideration, which was desired to be proved.

<u>LEMMA 3.</u> There exists a meromorphic function $S(z) = S(z, x, \theta)$ with the following properties: $S(z) > 0$ for $z > 0$;

$$|S(z)| < A/|z|, \; z \notin D = \{z: |\arg z| < \theta/2\}, \; A > 0; \tag{2.18}$$

$$|S(z)| \to \infty, \; |z| \to \infty \tag{2.19}$$

uniformly with respect to arg z in the closed domain $\bar{D}'$, $D' = D + 2$;

$$N(r, S) = \frac{x}{\pi\rho}\left(\cos\left(\frac{\rho\theta}{2}\right)\right) r^\rho + O(\log r), \; r \to \infty, \tag{2.20}$$

and all the poles lie on the positive ray;

$$\log|S(re^{i\varphi})| = xr^\rho \sin(\rho(\theta/2 - |\varphi|)^+) + O(\log r) \tag{2.21}$$

for $r \to \infty$ uniformly with respect to φ in each angle of the form $0 < \varepsilon < |\varphi| \leqslant \theta/2$, $\varepsilon > 0$.

Moreover, if $\theta = \pi/\rho$, then S is an entire function and

$$\log S(z) = xz^\rho + O(\log |z|), \quad |z| \to \infty, \tag{2.22}$$

uniformly with respect to $\varphi = \arg z$ in each angle of the form $|\varphi| \leqslant \alpha < \theta/2$. Here $z^\rho > 0$ for $z > 0$.

Proof. The charge, corresponding to the function $H(z-1)$, is concentrated on the three rays ℓ_j, $j = -1, 0, 1$, numbered in the anticlockwise direction. Let $\mu_j(t)$ denote the charge on the segment of the ray ℓ_j with the initial point at the point 1 and with length t. Let us set $X = \{z\colon \operatorname{dist}(z, \cup \ell_j) > \sin\theta/2\}$. Let us consider the function

$$u(z) = \sum_{j=-1}^{1} \int_0^\infty \log\left|1 - \frac{z}{1 + t\exp(ij\theta/2)}\right| d(\mu_j(t) - [\mu_j(t)]).$$

By Lemma 2, $|u(z)| = O(\log |z|)$, $z \in X$, $z \to \infty$. The function $S^*(z) = H(z-1) - u(z)$ has integral Riesz measure and can, therefore, be expressed in the form $\log |S_1(z)|$, where S_1 is a meromorphic function. The equality $\log |S_1(z)| = O(\log |z|)$ is valid in $\mathbf{C}\backslash D \subset X$. Dividing S_1 by a sufficiently high power of z, we get the desired function S with the property (2.18). The remaining properties (2.19)-(2.22) are obvious.

3. Let us consider the functions

$$S_k(z) = S(z\exp(-i\psi_k) - r_k, x_k, \theta_k),$$
$$S_k^*(z) = S(z\exp(i\psi_k) - r_k, x_k, \theta_k),$$

where $k \geqslant M$, and (x_k), (θ_k), and (ψ_k) are given sequences with the properties (2.1)-(2.4) and (2.10). We choose the numbers $r_k > 0$ so large that

$$|S_k(z)| < 2^{-2-k}, \quad z \notin D_k, \quad k \geqslant M; \tag{2.23}$$
$$N(r, S_k) \leqslant (\pi\rho)^{-1} x_k \cos(\rho\theta_k/2) r^\rho, \quad r > 0; \tag{2.24}$$
$$m(r, S_k) \leqslant (\pi\rho)^{-1} x_k (1 - \cos(\rho\theta_k/2)) r^\rho; \quad r > 0. \tag{2.25}$$

Such a choice of r_k is possible for (2.23) by virtue of (2.18), for (2.24) by virtue of (2.20), and for (2.25) by virtue of (2.21). Considering (2.19) and (2.23), we can increase r_k such that the set

$$Y_k = \{z\colon |S_k(z)| \in [2^{-1}, 2]\} \tag{2.26}$$

has small logarithmic area, i.e.,

$$\iint_{Y_k} r^{-1} dr d\theta < 2^{-k}. \tag{2.27}$$

Let us now set

$$h_1(z) = \sum_{k=M}^{\infty} (S_k(z) + S_k^*(z)).$$

The series is uniformly convergent in $\mathbf{C}$ by virtue of (2.23). The meromorphic function h_1 has the properties

$$|h_1(z)| \leqslant 1, \quad \arg z = \pm\varphi_k, \quad k \geqslant M, \tag{2.28}$$

by virtue of (2.23). The set

$$Y = \{z : 3/4 \leqslant |h_1(z)| \leqslant 7/4\} \subset \bigcup_{k=M}^{\infty} Y_k \tag{2.29}$$

has finite logarithmic area by virtue of (2.27). Further, if x is a number from (2.1), then

$$\log h_1(re^{i\varphi}) = x(re^{i(\varphi - \psi_M)})^\rho + O(\log r) \tag{2.30}$$

uniformly with respect to φ for $|\varphi - \psi_M| < \theta_M/3$, $r \to \infty$;

$$\log h_1(re^{i\varphi}) = x(re^{i(\varphi + \psi_M)})^\rho + O(\log r) \tag{2.31}$$

uniformly with resepct to φ for $|\varphi + \psi_M| < \theta_M/3$, $r \to \infty$. These relations follow from $\theta_M = \pi/\rho$. Consequently, S_M and S_M^* are entire functions and a formula, analogous to (2.22), is valid for them.

We have

$$\log|h_1(re^{i\varphi})| \sim x_k r^\rho \sin(\rho(\theta_k/2 - ||\varphi| - \psi|)) \tag{2.32}$$

for $r \to \infty$ uniformly with respect to φ in angles of the form $0 < \varepsilon < ||\varphi| - \psi_k| < \theta_k/2 - \varepsilon$, $k \geqslant M$. This follows from (2.21) and (2.23).

LEMMA 4. Let E_k be arbitrary angles of the form

$$\{z: |\arg z - \varphi_k| < \varepsilon\},\ \varepsilon < \frac{1}{2}\min(\theta_k, \theta_{k+1});$$

and E_k^* be the angles symmetric to E_k with respect to the real axis. The following asymptotics are valid for each $a \in \mathbf{C}$:

$$N(r, a, E_k, h_1) \sim N(r, a, E_k^*, h_1) \sim \begin{cases}(2\pi\rho)^{-1}(x_k + x_{k+1}) r^\rho, & r \to \infty,\ k \geqslant M;\\ (2\pi\rho)^{-1} x_M r^\rho, & r \to \infty,\ k = M-1;\end{cases} \tag{2.33}$$

$$N(r, a, D_k \backslash (E_{k-1} \cup E_k), h_1) + N(r, a, D_k^* \backslash (E_{k-1}^* \cup E_k^*), h_1) = O(\log r),\ r \to \infty,\ k \geqslant M; \tag{2.34}$$

$$N(r, \infty, D_k) \sim N(r, \infty, D_k^*, h_1) \sim (\pi\rho)^{-1} x_k r^\rho \cos\frac{\rho\theta_k}{2},\ r \to \infty,\ k \geqslant M. \tag{2.35}$$

Moreover, for each $\varepsilon > 0$ there exists a natural number K such that for each $a \in \mathbf{C}$

$$N\left(r, a, \overline{\bigcup_{k=K+1}^{\infty}(D_k \cup D_k^*)}, h_1\right) \leqslant \varepsilon r^\rho,\ r > r_0(a). \tag{2.36}$$

Proof. The relation (2.34) follows from (2.32), and (2.35) follows from (2.20). Further, for each $a \in \mathbf{C}$ the function $h_1(z) - a$ is a function of completely regular growth in the Levin–Pfluger sense [17] in the angles E_k, $k \geqslant M - 1$, and $E = \{z: -\varphi_{M-1} < \arg z < \varphi_{M-1}\}$. For the angles E_k, $k \geqslant M$, this follows from (2.32) and for the angle E this follows from a theorem of Cartwright [17, Chap. IV, Sec. 2, Theorem 6], since the indicator of the function h_1 is identically equal to 0 in E. Together with (2.32) for k = M, this gives complete regular growth in E_{M-1}. The indicator of h_1 is equal to

$$\left.\begin{matrix} x_{k+1}\sin\rho(\varphi - \varphi_k), & \varphi \geqslant \varphi_k,\\ x_k \sin\rho(\varphi_k - \varphi), & \varphi < \varphi_k,\end{matrix}\right\} re^{i\varphi} \in E_k,\ k \geqslant M.$$

If k = M − 1, then the second row must be replaced by zero. Hence, as we know, (2.33) follows [17].

To prove (2.36), let us, at first, find an upper bound for $N(r, a, h_1)$ with the help of (2.24), (2.25), and the inequality $|h_1(z) - S_k(z)| < 1/4$, $z \in D_k$:

$$N(r, a, h_1) \leqslant T(r, h_1) + O(1) = m(r, h_1) + N(r, h_1) + O(1) \leqslant 2\sum_{k=M}^{\infty} N(r, S_k) + \frac{1}{\pi}\sum_{k=M}^{\infty}\int_{re^{i\varphi}\in D_k^1} \log^+|h_1(re^{i\varphi})|\,d\varphi + O(1) \leqslant$$

$$\leqslant 2\sum_{k=M}^{\infty} N(r, S_k) + 2\sum_{k=M}^{\infty} m(r, S_k) + O(1) \leqslant 2(\pi\rho)^{-1} r^\rho \sum_{k=M}^{\infty} x_k \cos(\rho\theta_k/2) +$$

$$+ 2(\pi\rho)^{-1} r^\rho \sum_{k=M}^{\infty} x_k(1 - \cos(\rho\theta_k/2)) + O(1) = 2(\pi\rho)^{-1} r^\rho \sum_{k=M}^{\infty} x_k + O(1).$$

On the other hand, by virtue of (2.33) we have

$$N\left(r, a, \overline{\bigcup_{k=M-1}^{K}(D_k \cup D_k^*)}, h_1\right) \geqslant (2 + o(1))(\pi\rho)^{-1} r^\rho \sum_{k=M}^{K-1} x_k.$$

Consequently, (2.36) is fulfilled. The lemma is proved.

4. We will now carry out a quasiconformal deformation of h_1. The following lemma is easily proved.

LEMMA 5. For arbitrary $a \in \overline{\mathbf{C}}$, $|a| > 4$, there exists a quasiconformal mapping q_a of the "disk" $\{z: 5/4 \leqslant |z| \leqslant \infty\}$ onto itself that is the identity mapping on the circle $\{z: |z| = 5/4\}$ and is conformal for $|z| > 6/4$; $q_a(\infty) = a$, and the characteristic p_{q_a} is bounded by a constant that does not depend on a.

Let us now consider the set $\{z: |h_1(z)| > 5/4\}$. By virtue of (2.28), this set is decomposed into connected components that lie entirely in D_k and D_k^*, $k \geqslant M$. It is easily shown that exactly one unbounded component of this set lies in each angle D_k and D_k^*; we denote it by $\mathscr{B}_k$ $(\mathscr{B}_k^*)$. Let us define a new function:

$$h_2(z) = \begin{cases} h_1(z), \; z \in \mathbf{C} \setminus \bigcup_{k=M}^{\infty} (\mathscr{B}_k \cup \mathscr{B}_k^*), \\ q_{b_k}(h_1(z)), \; z \in \mathscr{B}_k \cup \mathscr{B}_k^*, \; k \geqslant M, \end{cases}$$

where q_{b_k} is the quasiconformal mapping of Lemma 5 and b_k is an element of the given sequence (b_k) with property (2.7). If $b_k = \infty$, then we assume that q_{b_k} is the identity mapping. It is obvious that h_2 is a pseudomeromorphic function. It is meromorphic everywhere, except the set $Y^* = \{z: 5/4 \leqslant |h_1(z)| \leqslant 6/4\}$, and has bounded characteristic on this set. Since the logarithmic area of the set Y^* is finite by virtue of (2.29), the function h_2 satisfies the condition (1.8). The following properties of h_2 follow from (2.33)-(2.35) and (2.32):

$$N(r, a, E_k, h_2) \sim N(r, a, E_k^*, h_2) \sim \\ \sim \begin{cases} (2\pi\rho)^{-1} r^{\rho} (x_k + x_{k+1}), \; r \to \infty, \; k \geqslant M, \\ (2\pi\rho)^{-1} r^{\rho} x_M, \; r \to \infty, \; k = M - 1, \end{cases} \tag{2.37}$$

where $a \neq b_k$, b_{k+1};

$$N(r, a, D_k \setminus (E_k \cup E_{k-1}), h_2) + \\ + N(r, a, D_k^* \setminus (E_k^* \cup E_{k-1}^*), h_2) = O(\log r), \; r \to \infty, \; a \neq b_k, \; k \geqslant M; \tag{2.38}$$

$$\log |h_2(re^{i\varphi}) - b_k|^{-1} \sim x_k r^{\rho} \sin(\rho((\theta_k/2) - ||\varphi| - \psi_k|)^+) \tag{2.39}$$

for $r \to \infty$ uniformly with respect to φ in the angles $0 < \varepsilon < ||\varphi| - \psi_k| < \theta_k/2 - \varepsilon$.

We show that for each $\varepsilon > 0$ there exists a natural number K such that

$$N\left(r, a, \overline{\bigcup_{k=K+1}^{\infty} (D_k \cup D_k^*)}, h_2\right) \leqslant \varepsilon r^{\rho}, \; r > r_0(a) \tag{2.40}$$

for almost all $a \in \mathbf{C}$. If $|a| < 5/4$, then (2.40) follows from (2.36). Then it follows from (2.40), (2.37), and (2.38) that

$$N(r, a, h_2) = (2 + o(1))(\pi\rho)^{-1} r^{\rho} \sum_{k=M}^{\infty} x_k, \; r \to \infty \tag{2.41}$$

for $|a| < 5/4$. Let us now observe that by virtue of the Teichmüller–Belinskii theorem we have $h_2 = f \circ \varphi$, where f is a meromorphic function and $\varphi(z) \sim z$, $z \to \infty$. Therefore, (2.41) is valid with h_2 replaced by f for $|a| < 5/4$. Hence by the Valiron theorem [1, Chap. IV, Sec. 2] we have (2.41) with h_2 replaced by f for almost all $a \in \mathbf{C}$. Consequently, (2.41) is valid for almost all $a \in \mathbf{C}$. Hence, again using (2.37) and (2.38), we get (2.40).

Let us observe that the quasiconformal deformation, constructed by us, does not affect the angles D_M and D_M^*, since $b_M = \infty$ (see (2.5)). In these angles we make one more deformation, as a result of which the asymptotic equations (2.30) and (2.31) turn into exact equations on the rays $\{z: \arg z = \pm\psi_M\}$.

LEMMA 6. Let the following analytic function be defined in the domain $D = \{z: |\arg z| < \theta\}$, $\theta < \pi/(2\rho)$:

$$f(z) = cz^{\rho} + O(\log |z|), \quad c > 0, \quad z \to \infty.$$

Then there exists a quasiconformal mapping β that is continuous and univalent in the closure of the domain $D' = \{z: |z| > r_0, \; 0 < \arg z < \theta/2\}$, has the property (1.8), and fulfills the conditions

$$\beta(z) = z, \; \arg z = \theta/2, \; |z| > r_0;$$
$$f(\beta(z)) = cr^{\rho}, \; r > r_0.$$

Proof. It is sufficient to prove the lemma for $c = \rho = 1$. Let us map the sector $D \setminus \{z: |z| \leqslant 1\}$ onto the halfstrip $\Pi = \{\xi + i\eta: \; \xi > 0, \; |\eta| < 2\}$ by means of the function $\zeta = \chi(z) = 2\theta^{-1} \log z$. We set $h(\zeta) = \chi \circ f \circ \chi^{-1}(\zeta) = \zeta + O(\exp(-c\zeta))$, $c > 0$, $\xi = \mathrm{Re}\zeta \to +\infty$. The function h is univalent in a halfstrip $\Pi' = \{\xi + i\eta: \ \xi > r_1, \ |\eta| < 1\}$. It is obvious that the image $h(\Pi')$ contains a halfstrip $\Pi'' = \{\xi = i\eta: \ \xi > r_2, \ |\eta| < 1/2\}$, and the inverse function satisfies the following conditions in Π'':

$$h^{-1}(\zeta)=\zeta+O(\exp(-c_1\zeta)),$$
$$(h^{-1})'(\zeta)=1+O(\exp(-c_1\zeta)), \ \operatorname{Re}\zeta\to+\infty.$$

Let $\Gamma \subset \Pi'$ denote the inverse image of the ray $\{\zeta: \ \zeta > r_2\}$ under mapping by the function h. It is easily seen that the curve Γ, starting from a certain place, is the graph of a certain function $\eta = \gamma(\xi)$, $\xi > r_3$, such that $\gamma(\xi) = O(\exp(-c_2\xi)$,

$$\gamma'(\xi)=O(\exp(-c_2\xi)), \ |\gamma(\xi)|<1/2,$$
$$\gamma(\operatorname{Re} h^{-1}(\xi))=\operatorname{Im} h^{-1}(\xi). \tag{2.42}$$

Let us consider the quasiconformal mapping α_1, defined in the halfstrip $\Pi_1 = \{\xi + i\eta: \ \xi > r_3, \ 0 < \eta < 1\}$ as follows:

$$(\xi,\eta)\mapsto(\xi,\gamma(\xi)+\eta(1-\gamma(\xi))).$$

The characteristic p of this mapping is easily estimated: $p = 1 + O(\sqrt{\gamma^2 + (\gamma')^2}$, $\xi \to +\infty$. It is obvious that

$$\iint_{\Pi_1}(p(\zeta)-1)\,d\xi d\eta<\infty. \tag{2.43}$$

Further, let us consider the mapping α_2: $\Pi_1 \to \Pi_1$, defined as follows: $(\xi, \eta) \to (\xi\eta + (\operatorname{Re} h^{-1}(\xi))(1 - \eta), \eta)$. The characteristic of this mapping also satisfies the condition (2.43), since

$$\xi\eta+(\operatorname{Re} h^{-1}(\xi))(1-\eta)=\xi+\delta(\xi),$$
$$\delta(\xi)=O(\exp(-c_1\xi)), \ \delta'(\xi)=O(\exp(-c_1\xi)), \ \xi\to+\infty.$$

Let us set $\alpha = \alpha_1 \circ \alpha_2$. By virtue of (2.42), we have $\alpha(\zeta) = \zeta$ for $\operatorname{Im}\zeta = 1$ and $\operatorname{Re}\zeta > r_3$ and $h(\alpha(\zeta)) = \zeta$ for $\operatorname{Im}\zeta = 0$ and $\operatorname{Re}\zeta > r_3$. The characteristic of the mapping α satisfies the condition (2.43). Let us extend α by the identity mapping in the halfstrip $\{\xi + i\eta: \ \xi > r_3, \ 0 < \eta < 2\}$ and set $\beta = \chi^{-1}\circ\alpha\circ\chi$. The mapping β is the desired one. The inequality (1.8) follows from (2.43).

Using Lemma 6 and the relations (2.30) and (2.31), we make quasiconformal deformation in the angles $\{z: \ \psi_M < \arg z < \varphi_M\}$ and $\{z: \ -\varphi_M < \arg z < -\psi_M\}$ such that the new function (for which we retain the old symbol h_2) is pseudomeromorphic for $\psi_M < \arg z < 2\pi - \psi_M$ and, besides (2.37)-(2.40), we have

$$h_2(r\exp(\pm i\psi_M))=\exp xr^\rho, \ r>r_0. \tag{2.44}$$

5. Now we can complete the construction of f. We construct the function h_3 in the same manner as h_2, but h_3 is pseudomeromorphic in the angle $-\psi_1 < \arg z < \psi_1$ and has the following properties (cf. (2.37)-(2.40), (2.44)):

$$N(r,a,E_k,h_3)\sim N(r,a,E_k^*,h_3)\sim(2\pi\rho)^{-1}r^\rho(x_k+x_{k+1}),$$
$$r\to\infty, \ k\leqslant 0, \ a\notin\{b_j\}; \tag{2.45}$$
$$N(r,a,D_k\setminus(E_k\cup E_{k-1}),h_3)+N(r,a,D_k^*\setminus(E_k^*\cup E_{k-1}^*),h_3)=O(\log r),$$
$$r\to\infty, \ k\leqslant 0, \ a\neq b_k; \tag{2.46}$$
$$\log|h_3(re^{i\varphi})-b_k|^{-1}\sim x_k r^\rho\sin(\rho(\theta_k/2-||\varphi|-\psi_k|)^+) \tag{2.47}$$

for $r \to \infty$ uniformly with respect to φ in angles of the form $0<\varepsilon<||\varphi|-\psi_k|<\theta_k/2-\varepsilon$. For each $\varepsilon > 0$ there exists a natural number K such that for almost all $a\in\mathbb{C}$

$$N\left(r,a,\overline{\bigcup_{k=K+1}^{\infty}(D_{-k}\cup D_{-k}^*)},h_3\right)\leqslant\varepsilon r^\rho, \ r>r_0(a); \tag{2.48}$$

$$h_3(r\exp(\pm i\psi_1))=\exp(-xr^\rho), \ r>r_0. \tag{2.49}$$

Let us now define the function g for $|z| > r_0$:

$$g(z)=\begin{cases} g_1(z), & \psi_1\leqslant \arg z<\psi_M,\\ h_2(z), & \psi_M\leqslant \arg z<2\pi-\psi_M,\\ g_1^*(z), & -\psi_M\leqslant \arg z<-\psi_1,\\ h_3(z), & -\psi_1\leqslant \arg z<\psi_1.\end{cases}$$

By virtue of (2.15), (2.16), the properties of the function g_1^* indicated at the end of p. 1, (2.44), and (2.49), the function g is continuous for $|z| \geqslant r_0$. It is also obvious that it is pseudomeromorphic with the property (1.8). We change the function g in a bounded domain and define it such that it becomes pseudomeromorphic in $\mathbf{C}$ [1, Chap. VII].

For arbitrary $a\in\overline{\mathbf{C}}$ we set

$$N(a, E_k)=N\left(a, E_k^*\right)=\lim_{r\to\infty}\pi\rho r^{-\rho}N(r, a, E_k, g).$$

By virtue of (2.40), (2.48), (2.13), (2.38), (2.46), and the definition of the function g_1^*, for almost all $a\in\mathbf{C}$ we have

$$N(r, a, g)=(2+o(1))(\pi\rho)^{-1}r^{\rho}\sum_{k=-\infty}^{\infty}N(a, E_k),\quad r\to\infty. \tag{2.50}$$

If $a\notin\{b_j\}$, then, by (2.12), (2.1), (2.37), (2.45), and (2.3),

$$\sum_{k=-\infty}^{\infty}N(a, E_k)=\frac{M-1}{2}x+\frac{1}{2}\sum_{k=M}^{\infty}(x_k+x_{k+1})+\frac{1}{2}\sum_{k=0}^{\infty}(x_{-k}+x_{-k+1})=\frac{x_1}{2}+\sum_{j=1}^{M/2}x_{2j}+\sum_{k=M+1}^{\infty}x_k+\sum_{k=0}^{\infty}x_{-k}=1. \tag{2.51}$$

Since g satisfies the condition (1.8), then exists a meromorphic function f such that $g = f\circ\varphi$, $\varphi(z)\sim z$, $z\to\infty$. If $a\in\mathbf{C}$ does not belong to a certain exceptional set of measure zero, then by virtue of (2.50) and (2.51) we have

$$T(r, f)\sim N(r, a, f)\sim N(r, a, g)\sim 2(\pi\rho)^{-1}r^{\rho},\ r\to\infty,$$

i.e., (2.8). In particular, the function f has order $\rho < \infty$.

Let ε be an arbitrary positive number. By a theorem of Edrei and Fuchs [1, Chap. I, Theorem 7.3] there exists a $\tau > 0$ such that for each set $E_\tau \subset [0, 2\pi]$ of length τ and $a\in\overline{\mathbf{C}}$ arbitrary

$$(2\pi)^{-1}\int_{E_\tau}\log^+|f(re^{i\varphi})-a|^{-1}d\varphi\leqslant\varepsilon r^{\rho},\quad r>r_0(a).$$

We choose a finite union of open intervals that cover all the points $0, \pi, \pm\varphi_k, \pm\psi_k, k\in\mathbf{Z}$, as E_τ. Then the uniform asymptotics (2.14), (2.39), and (2.47) are valid for $\varphi\notin E_\tau$. Therefore, for arbitrary $a\in\overline{\mathbf{C}}$

$$m(r, a, f)=(2+o(1))(\pi\rho)^{-1}r^{\rho}\sum_{\{k:\, b_k=a\}}x_k\left(1-\cos\frac{\rho\theta_k}{2}\right)+\alpha(r),$$

where $|\alpha(r)|\leqslant\varepsilon r^{\rho}$. Dividing by r^ρ and taking limit, at first, as $r \to \infty$ and then as $\varepsilon \to 0$, we get (2.9).

Finally, (2.11) follows at once from (2.14), (2.39), and (2.47). Thus, a function with the properties (2.8), (2.9), and (2.11) has been constructed.

3. Proof of Theorem 1. Without loss of generality, we can assume that $a_1 = \infty$, $a_2 = 0$, $\delta_i \geqslant \delta_j$ for $i < j$, and the annulus $\{z: 4 < |z| < 5\}$ contains infinite set of the numbers a_j. All this can be achieved by making a bilinear transformation and renumbering a_j. Let us construct a two-sided sequence (b_j) with the properties (2.5)-(2.7), the set of whose values coincides with $\{a_j\}$ (and all the sets $\{j: b_j = a_k\}$ are finite), and a sequence of positive numbers (d_j), $j\in\mathbf{Z}$, with the following properties:

$$\sum_{\{j:\, b_j=a_k\}}d_j=\delta_k; \tag{3.1}$$

$$d_1=d_2=\ldots=d_M=M_0^{-1},\quad M_0>0; \tag{3.2}$$

$$\frac{d_1}{2}+\sum_{j=0}^{\infty}d_{-j}+\sum_{j=M+1}^{\infty}d_j+\sum_{j=1}^{M/2}d_{2j}<1. \tag{3.3}$$

To prove the possibility of construction of these sequences, let us consider two cases.

1st Case. $\delta_1 < \Delta/2$. We choose a natural number N so large that

$$\sum_{j=1}^{N} \delta_j > 2\delta_1, \tag{3.4}$$

$$\sum_{j=N+1}^{\infty} \delta_j < (2-\Delta)/8. \tag{3.5}$$

Next, we approximate the numbers $\delta_1, \ldots, \delta_N$ by smaller rational numbers δ_j^* such that

$$\delta_j^* = M_j/M_0, \quad 1 \leqslant j \leqslant N, \quad M_j > 0 \ \text{ even}, \quad M_0 > 8/(2-\Delta); \tag{3.6}$$

$$\delta_j = \delta_j^* + \delta_j', \quad \delta_j' > 0, \quad 1 \leqslant j \leqslant N; \tag{3.7}$$

$$\sum_{j=1}^{N} \delta_j' < (2-\Delta)/4; \tag{3.8}$$

$$\sum_{j=1}^{N} \delta_j^* > 2\delta_k^*, \quad 1 \leqslant k \leqslant N. \tag{3.9}$$

The condition (3.9) is fulfilled by virtue of (3.4). The relations (3.6)-(3.8) are obtained if we choose the number M_0 sufficient large and set $M_j = 2[(1/2)M_0\delta_j] + 2$. Let $M = \sum_{j=1}^{N} M_j$, and define d_j for $1 \leqslant j \leqslant M$ by Eq. (3.2). Let us define the numbers b_1 and b_M as required in (2.5). We choose the numbers $b_2, \ldots, b_{M-1}$ such that the set $b_1, \ldots, b_M$ contains precisely M_j numbers equal to a_j, $1 \leqslant j \leqslant N$. The inequality (3.9) implies that $M > 2M_j$. Therefore the numbers $b_1, \ldots, b_M$ can be ordered such that (2.6) holds.

We decompose the sequence $\delta_1', \delta_2', \ldots, \delta_N', \delta_{N+1}, \ldots$ into two infinite parts such that $|a_j| > 4$ for the numbers j of the first part and $|a_j| < 5$ for the numbers j of the second part. We enumerate the first part as a subseries with the natural numbers, starting from $M + 1$, and the second part with all nonositive integers. We get a sequence (d_j), $j \geqslant M + 1$ and $j \leqslant 0$. If $d_j = \delta_k$ or δ_k', then we set $b_j = a_k$. Thus, the sequences (b_j) and (d_j) are constructed. The properties (3.1), (3.2), and (2.5)-(2.7) are valid by construction. To prove (3.3) we use (3.2), (3.6), (3.5), and (3.8) successively:

$$\frac{d_1}{2} + \sum_{j=0}^{\infty} d_{-j} + \sum_{j=M+1}^{\infty} d_j + \sum_{j=1}^{M/2} d_{2j} \leqslant \frac{2-\Delta}{16} + \frac{2-\Delta}{8} + \frac{2-\Delta}{4} + \frac{1}{2}\sum_{j=1}^{N} \delta_j^* < \frac{2-\Delta}{2} + \frac{\Delta}{2} = 1.$$

2nd Case. $\delta_1 \geqslant \Delta/2$. We choose a number N so large that

$$\sum_{j=N+1}^{\infty} \delta_j < (1-\delta_1)/4. \tag{3.10}$$

Now we choose even numbers M_j, $0 \leqslant j \leqslant N$, such that (3.6) and (3.7) with $2 \leqslant j \leqslant N$ are fulfilled and, moreover,

$$M_0 > 4/(1-\delta_1), \quad \sum_{k=2}^{N} \delta_k' < (1-\delta_1)/8. \tag{3.11}$$

Let us set

$$M = 2\sum_{k=2}^{N} M_k, \tag{3.12}$$

$$\delta_1^* = \sum_{k=2}^{N} \delta_k^*. \tag{3.13}$$

Then

$$\delta_1 = \delta_1^* + \delta_1', \quad \delta_1' > 0, \tag{3.14}$$

since $\delta_1 > \sum_{j=2}^{\infty} \delta_j > \sum_{j=2}^{N} \delta_j^*$. Let us now define the numbers $d_1, \ldots, d_M$ by Eq. (3.2). We set $b_1 = a_2$ (=0) and $b_{2j} = a_1 (=\infty)$, $1 \leqslant j \leqslant M/2$. We choose the numbers b_j with odd indices j,

$1 \leqslant j < M$, such that they include precisely M_k numbers equal to a_k, $1 \leqslant k \leqslant N$. This is possible by virtue of (3.12). We deal with the numbers δ_k', $1 \leqslant k \leqslant N$, defined in (3.7) and (3.14), and the numbers δ_k, $k \geqslant N + 1$, in exactly the same manner as in the first case. We get sequences (b_j) and (d_j), $j \in \mathbf{Z}$, with the properties (3.1), (3.2), and (2.5)-(2.7).

Let us verify (3.3). By virtue of (3.11), (3.10), (3.13), and (3.14), we have

$$\frac{d_1}{2} + \sum_{j=0}^{\infty} d_{-j} + \sum_{j=M+1}^{\infty} d_j + \sum_{j=1}^{M/2} d_{2j} < \frac{1-\delta_1}{8} + \delta_1' + \frac{1-\delta_1}{4} + \frac{1-\delta_1}{8} + \sum_{j=2}^{N} \delta_j^* = \frac{1-\delta_1}{2} + \delta_1' + \delta_1^* = \frac{1}{2} + \frac{\delta_1}{2} < 1.$$

Thus, sequences with the properties (3.1)-(3.3) and (2.5)-(2.7) have been constructed. It follows from (1.5) that

$$\sum_{j=-\infty}^{\infty} d_j^{1/3} < \infty. \tag{3.15}$$

LEMMA 7. Let there be given a sequence (d_j), $j \in \mathbf{Z}$, with the properties (3.2), (3.3), and (3.15). Let us set $A = \mathbf{Z} \setminus \{1, 3, 5, \ldots, M-1\}$. Then there exist sequences (x_k) and (θ_k), $k \in \mathbf{Z}$, and a number $\rho > 1$ such that

$$0 < \theta_k \leqslant \pi/\rho, \ x_k > 0, \ k \in \mathbf{Z}; \tag{3.16}$$

$$x_1/2 + \sum_{k \in A} x_k = 1, \quad \sum_{k \in \mathbf{Z}} \theta_k = \pi; \tag{3.17}$$

$$x_1 = x_2 = \ldots = x_M, \ \theta_1 = \theta_2 = \ldots = \theta_M = \pi/\rho; \tag{3.18}$$

i.e., (2.1)-(2.4) are fulfilled, and, moreover,

$$d_k = (1 - \cos(\rho\theta_k/2))x_k, \ k \in \mathbf{Z}. \tag{3.19}$$

Proof. At first, we choose $N > M$ such that

$$\sum_{|k|>N} d_k^{1/3} < 1 - \sum_{k \in A} d_k - \frac{d_1}{2}, \tag{3.20}$$

which is possible by virtue of (3.3) and (3.15). Let us set

$$x_k = d_k, \ z_k = 1, \ k \in A, \ |k| \leqslant N; \tag{3.21}$$

$$x_k = t^2 d_k^{1/3}, \ z_k = t^{-1} d_k^{1/3}, \ |k| > N, \tag{3.22}$$

where $t \geqslant 1$ is a parameter, which will be fixed later on. For each $t \geqslant 1$ we have

$$x_k z_k^2 = d_k, \ k \in A. \tag{3.23}$$

The sum

$$S(t) = x_1/2 + \sum_{j \in A} x_j = \frac{d_1}{2} + \sum_{\substack{j \in A \\ |j| < N}} d_j + t^2 \sum_{|j|>N} d_j^{1/3} \to +\infty, \ t \to \infty,$$

is a continuous increasing function of t and $S(1) < 1$ by virtue of (3.20). Therefore, we can fix a value of $t > 1$ such that the first equation of (3.17) is valid. Let us observe that $0 < z_k \leqslant 1$ by virtue of (3.21), (3.22), and (3.20). Let y_k denote the solution of the equation

$$1 - \cos y = z_k^2, \ 0 < y_k \leqslant \pi/2. \tag{3.24}$$

It follows from (3.15) and (3.22) that the series $\sum_{k=-\infty}^{\infty} y_k$ is convergent. Let us denote the sum of this series by $\pi\rho/2$. Let us now set $\theta_k = 2y_k/\rho$. Then, by virtue of the choice of ρ, the second equation of (3.17) is valid and, by virtue of (3.24), the inequality (3.16) holds. Finally, (3.18) follows from (3.21) and (3.24), and (3.19) is none else than (3.23). The lemma is proved.

The function f of Sec. 2 with the selected values of the parameters is the desired one. Indeed, it follows from (3.8), (3.9), (3.19), and (3.1) that $\delta(a_j, f) = \delta_j$ for $j \in \mathbf{N}$ and $\delta(a, f) = 0$ for $a \notin \{a_j\}$. The order of f is finite and is equal to ρ.

4. Proof of Theorem 2. Without loss of generality, we can assume that the annulus $\{z: 4 < |z| < 5\}$ contains an infinite set of the numbers a_j. The case of finite ω is obtained by a simple modification of the proof. Moreover, we assume that $0, \infty \in \{a_j\}$, $\beta(\infty) \leqslant \beta(0)$. Let us consider a sequence (b_j), $j \in \mathbf{Z}$, the set of whose values coincides with $\{a_j\}$ and each nonzero number a_j occurs once and the number 0 occurs twice among (b_j), $b_1 = 0$, $b_2 = \infty$, $|b_j| > 4$ for $j \geqslant 2$, and $|b_j| < 5$ for $j \leqslant 1$. Let us define the sequence (d_j) as follows: If $b_j = a_k$, $j \neq 1$, then $d_j = \beta_k$ and, moreover, $d_1 = \beta(\infty)$. Thus, $d_1 = d_2$.

LEMMA 8. Let there be given a sequence (d_j), $j \in \mathbf{Z}$, such that $d_j > 0$, $\sum_{j=-\infty}^{\infty} d_j^{1/2} < \infty$, and $d_1 = d_2$. Then there exist two sequences (x_j) and (θ_j), $j \in \mathbf{Z}$, and a number $\rho > 0$ such that (2.1)-(2.4) are fulfilled with $M = 2$ and

$$d_j = \frac{\pi\rho}{2} x_j \sin(\rho\theta_j/2), \quad j \in \mathbf{Z}. \tag{4.1}$$

Proof. For each natural number $N \geqslant 3$ such that $d_k < 1$ for $|k| > N$, we set

$$\left.\begin{aligned} x_k' &= d_k, \\ y_k' &= \pi/2 \end{aligned}\right\}, \ |k| \leqslant N; \qquad \left.\begin{aligned} x_k' &= \sqrt{d_k}, \\ y_k' &= \arcsin \sqrt{d_k} \end{aligned}\right\}, \ |k| > N.$$

If N is increased, then the sum $S_1 = \sum_{k=-\infty}^{\infty} x_k' - x_1'/2$ decreases and the sum $S_2 = \sum_{k=-\infty}^{\infty} y_k'$ increases unboundedly. We fix N such that $S_1 < S_2$. Let us set $x_k'' = t x_k'$ and $y_k = \arcsin(t^{-1} \sin y_k')$, where $t \geqslant 1$, $k \geqslant 3$, $x_1'' = x_2'' = x_1' = x_2'$, and $y_1 = y_2 = \pi/2$. For arbitrary $t \geqslant 1$ we have

$$d_j = x_j'' \sin y_j, \quad y_j \leqslant \pi/2, \quad j \in \mathbf{Z}. \tag{4.2}$$

As t increases, the sum $S_1(t) = \sum_{k=-\infty}^{\infty} x_k'' - x_1''/2$ increases unboundedly and the sum $S_2(t) = \sum_{k=-\infty}^{\infty} y_k$ decreases. Since $S_2(1) = S_2 > S_1 = S_1(1)$, we can find $t > 1$ such that $S_1(t) = S_2(t)$.

Let us now set $x_k = 2x_k''/(\pi\rho)$, where $\rho > 0$ is chosen such that (2.3) is fulfilled, i.e.,

$$\rho = \frac{2}{\pi} S_1(t) = \frac{2}{\pi} S_2(t).$$

Then $S_2(t) = \sum_{k=-\infty}^{\infty} y_k = \pi\rho/2$, and, setting $\theta_k = 2y_k/\rho$, we get (2.4). Finally, (4.1) and (2.2) follow from (4.2). The lemma is proved.

Using the constructed sequences (b_j), (x_j), and (θ_j) with the properties (2.1)-(2.7) for $M = 2$, we construct a meromorphic function f with the properties (2.8), (2.9), and (2.11). The order of this function is equal to ρ. It follows from (2.8), (2.11), and (4.1) that if $a_j = b_k$ for $k \neq 1$, then

$$\beta(a_j, f) = \varliminf_{r\to\infty} \frac{2}{\pi\rho} r^{-\rho} \log M(r, a, f) \geqslant \varliminf_{r\to\infty} \frac{2}{\pi\rho} r^{-\rho} \max_{|z|=r, z \in D_k} \log|f(z) - b_k|^{-1} = \frac{2}{\pi\rho} x_k \sin \rho\theta_k/2 = d_k = \beta_j.$$

In order to obtain the reverse inequality, we set $\beta(a) = \beta_j$ if $a = a_j$ and $\beta(a) = 0$ if $a \notin \{a_j\}$. Moreover, let $A(a) = \varnothing$, if $a \notin \{a_j\}$; $A(a) = [\varphi_{k-1}, \varphi_k]$, if $a = b_k \neq 0$, and $A(a) = [\varphi_{k-1}, \varphi_k] \cup [\varphi_0, \varphi_1]$, if $a = b_k = 0$, $k \neq 1$. It follows from (2.9) and (2.11) that

$$\int_{|\varphi| \notin A(a)} \log^+ |f(re^{i\varphi}) - a|^{-1} d\varphi = o(r^\rho), \quad r \to \infty, \quad a \in \overline{\mathbf{C}}.$$

In [18] the author has proved that this implies that

$$\sup_{|\varphi| \notin A(a)} \log^+ |f(re^{i\varphi}) - a|^{-1} = o(r^\rho),$$

for $r \to \infty$ outside a set of zero density. By construction, the function $f(z) - b_k$ is holomorphic in the angle $||\arg z| - \psi_k| < \theta_k/2 - \varepsilon$ for arbitrary $\varepsilon > 0$ and for each $k \in \mathbf{Z}$. It follows from (2.11) that this function has completely regular growth in the indicated angle. Hence

$$\sup_{|\varphi| \in A(a)} \log^+ |f(re^{i\varphi}) - a|^{-1} \leqslant (\beta(a) + o(1)) r^{\rho},$$

for $r \to \infty$ outside a set of zero density. Thus, $\beta(a, f) = \beta(a)$, which was desired to be proved.

The author thanks V. S. Azarin, A. A. Gol'dberg, and M. L. Sodin for advice and numerous remarks.

LITERATURE CITED

1. A. A. Gol'dberg and I. V. Ostrovskii, Distribution of Values of Meromorphic Functions [in Russian], Nauka, Moscow (1970).
2. D. Drasin, "The inverse problem of the Nevanlinna theory," Acta Math., 138, 83-151 (1977).
3. R. Nevanlinna, Le Théorème de Picard-Borel et la Théorie des Fonctions Méromorphes, Gauthier-Villars, Paris (1929).
4. W. Hayman, Meromorphic Functions, Clarendon Press, Oxford (1964).
5. A. A. Gol'dberg, "On the inverse problem of the value distribution theory for meromorphic functions," Ukr. Mat. Zh., 6, No. 4, 385-397 (1954).
6. O. Teichmüller, "Vermutungen und Sätze über die Vertverteilung gebrochener Functionen endlicher Ordnung," Deutsche Math., 4, 163-190 (1939).
7. A. Weitsman, "A theorem on Nevanlinna deficiencies," Acta Math., 128, 41-52 (1972).
8. A. Weitsman, "Meromorphic functions with maximal deficiency sum and conjecture of R. Nevanlinna," Acta Math., 123, 115-139 (1969).
9. D. Drasin, "Quasiconformal modifications of functions having deficiency sum two," Ann. Math., 114, 439-518 (1981).
10. R. Nevanlinna, "Über Riemannsche Flächen mit endlich vielen Windungspunkten," Acta Math., 58, 295-373 (1932).
11. N. U. Arakelyan, "Entire functions of finite order with infinite set of deficient values," Dokl. Akad. Nauk SSSR, 170, No. 5, 999-1002 (1966).
12. V. P. Petrenko, Growth of Meromorphic Functions [in Russian], Vishcha Shkola, Kharkov (1978).
13. A. É. Eremenko, "On deviations of meromorphic functions of finite lower order," in: Theory of Functions, Functional Analysis, and Their Applications [in Russian], No. 30, Vishcha Shkola, Kharkov (1983), pp. 56-64.
14. Le Van Thiem, "Über das Umkehrproblem der Vertverteilungslehre," Comment. Helv., 23, 77-86 (1949).
15. P. P. Belinskii, General Properties of Quasiconformal mappings [in Russian], Nauka, Novosibirsk (1974).
16. S. Stoilov, Lectures on the Topological Principles in the Theory of Analytic Functions [in Russian], Nauka, Moscow (1964).
17. B. Ya. Levin, Distribution of Zeros of Entire Functions, Amer. Math. Soc. (1972).
18. A. É. Eremenko, "On deficiencies and deviations of meromorphic functions of finite order," Dokl. Akad. Nauk Ukr. SSR, No. 1, 18-20 (1985).