quantum affine algebras of type A_n^1 (the Kats–Moody quantum algebras were introduced independently in [8, 1]).

The author thanks A. V. Zelevisnkii for useful advice.

LITERATURE CITED

1. V. G. Drinfel'd, "Hopf algebras and the quantum Yang–Baxter equation," Dokl. Akad. Nauk SSSR, 283, No. 5, 1060-1064 (1985).
2. I. N. Bernshtein and A. V. Zelevinskii, "Representations of the group GL(n, F), where F is a local Archimedean field," Usp. Mat. Nauk, 31, No. 3, 5-70 (1976).
3. J. D. Rogawski, "On modules over the Hecke algebra of a p-adic group," Invent. Math., 79, No. 3, 443-465 (1985).
4. G. Lusztig, "Some examples of square integrable representations of semisimple p-adic groups," Trans. Am. Math. Soc., 277, No. 2, 623-653 (1983).
5. A. Borel, "Admissible representations of a semisimple group over a local field with vectors fixed under an Twahori subgroup," Invent. Math., 35, 233-259 (1976).
6. H. Matsumoto, Analyse Harmonique dans les Systems de Tits Bornoloques de Type Affine, Lect. Notes in Math., Vol. 590, Springer-Verlag, Berlin–New York (1977).
7. M. Jimbo, RIMS Preprint No. 517, Kyoto Univ. (1985).
8. M. Jimbo, "A q-difference analogue of U(g) and the Yang–Baxter equation," Lett. Math. Phys., 10, No. 1, 63-69 (1985).

A HYPOTHESIS OF LITTLEWOOD AND THE DISTRIBUTION OF VALUES OF ENTIRE FUNCTIONS

A. É. Eremenko and M. L. Sodin

UDC 517.53

For a function f, meromorphic in **C**, we denote by ρ_f the spherical derivative $\rho_f(z) = |f'(z)|/(1+|f(z)|^2)$. Let $D(r)=\{z:|z|\leqslant r\}$, and let m_2 be Lebesgue measure in **C**. Following Littlewood [1], we consider the quantities $\varphi(n)=\sup_f \iint_{D(1)} \rho_f\, dm_2, n\in\mathbf{N}$, where the upper bound is taken over all polynomials f of degree n. We denote analogous quantities for rational functions by $\psi(n)$. It follows from the Schwarz–Bunyakovskii inequality that

$$\varphi(n)\leqslant\psi(n)\leqslant\Big(\iint_{D(1)} dm_2 \sup_f \iint_{D(1)} \rho_f^2\, dm_2\Big)^{1/2}\leqslant\pi\sqrt{n}.$$

The best known lower bounds were obtained by Hayman [2]: $\varphi(n)\geqslant A_1\log n$, $\psi(n)\geqslant A_2\sqrt{n}$. Here and in the sequel the A_k are absolute constants. In [1] it was conjectured that

$$\varphi(n)\leqslant A_3 n^{1/2-\alpha} \qquad (1)$$

for some $\alpha > 0$.

<u>THEOREM 1.</u> $\varphi(n)=o(\sqrt{n}),\ n\to\infty.$

From the hypothesis (1) Littlewood derived a remarkable result, which may be stated thus: For an arbitrary entire function f of finite nonzero order an infinitely small portion S of the plane can be found such that for almost all w the roots of the equation f(z) = w lie in S, with a negligible exception. The analysis of elliptic functions in [2] shows that this assertion is invalid if entire functions are replaced by meromorphic functions.

<u>Example.</u> f(z) = exp z. We can put $S=\{x+iy:|y|>x^2\}$. For an arbitrary w all the roots of the equation f(z) = w, with the exception of a finite number, belong to S. The set S has zero density, $m_2(S\cap D(r))=o(r^2)$, $r\to\infty$.

Physicotechnical Institute of Low Temperatures, Academy of Sciences of the Ukrainian SSR. Kharkov Institute of Radio-Electronics. Translated from Funktsional'nyi Analiz i Ego Prilozheniya, Vol. 20, No. 1, pp. 71-72, January-March, 1986. Original article submitted December 26, 1984.

 0016-2663/86/2001-0060$12.50

THEOREM 2. Let f be an entire function of finite order, and let $\lambda(r)$ be its proximate order. Then there exists a set $S \subset \mathbf{C}$ of zero density such that for an arbitrary $w \in \mathbf{C}$ the relation $n(r, w) = n_S(r, w) + o(r^{\lambda(r)})$, $r \to \infty$ is satisfied. Here $n_S(r, w)$ is the number of roots of the equation $f(z) = w$ in $S \cap D(r)$.

The proofs of Theorems 1 and 2 are based on an elementary lemma from potential theory, a particular case of which is contained in [3, 4].

LEMMA. Let $u \geqslant 0$ be a subharmonic function and let μ be its Riesz measure. Then $\{z: u(z) = 0\} = E \cup L$, where $\mu(E) = 0$, $m_2(L) = 0$.

As E we can take a set of points having the density of the set

Proof of Theorem 1. We assume that we can find an infinite set of numbers $\mathbf{N}_1$ and polynomials f_n, $\deg f_n = n \in \mathbf{N}_1$, such that

$$\iint\limits_{D(1)} \rho_{f_n}\, dm_2 \geqslant A_4 \sqrt{n}, \quad n \in \mathbf{N}_1. \tag{2}$$

We consider the family of subharmonic functions $v_n(z) = \frac{1}{n} \log \sqrt{1 + |f_n(z)|^2}$ with Riesz measures μ_n. A direct calculation shows that the Laplacian

$$\Delta v_n(z) = \frac{2}{n} \rho_{f_n}^2(z). \tag{3}$$

In particular, $\mu_n(\mathbf{C}) = 1$. Selecting a subsequence, we can assume that $\mu_n \to \mu$ weakly in each disk $D(r)$, $r > 0$, $n \in \mathbf{N}_2 \subset \mathbf{N}_1$. Two cases are possible

1°. $\liminf v_n < +\infty$. Selecting a subsequence, $\mathbf{N}_3 \subset \mathbf{N}_1$, we assume that $v_n \to u$ in the mean disk, $n \in \mathbf{N}_3$. Applying the lemma to the function $u \geqslant 0$, we obtain three sets M, L, E such that $u > 0$ on M, $\mu(E) = 0$, $m_2(L) = 0$, $D(1) = M \cup L \cup E$. We fix an $\varepsilon > 0$, sufficiently small. We select δ, $0 < \delta < \varepsilon$, so that the set $M = \{z \in D(1): u(z) \geqslant 2\delta\}$ will possess the property $m_2(M \setminus M') < \varepsilon$. Following this, we select a closed set $E' \subset E$ so that the inequality $m_2(E \setminus E') < \varepsilon$ is satisfied. It is obvious that $\mu(E') = 0$; therefore, for sufficiently large $n \in \mathbf{N}_3$, we have

$$\mu_n(E') < \varepsilon. \tag{4}$$

Let us put $L' = D(1) \setminus (E' \cup M')$. Then

$$m_2(L') < 2\varepsilon. \tag{5}$$

From the convergence of $v_n \to u$ it follows that sets L_n can be found such that

$$m_2(L_n) < \varepsilon \text{ and } v_n(z) \geqslant \delta \text{ for } \quad z \in M' \setminus L_n, \quad n \in \mathbf{N}_3. \tag{6}$$

For an arbitrary measurable set $T \subset D(1)$ the Schwarz–Bunyakovskii inequality yields

$$\iint\limits_T \rho_f\, dm_2 \leqslant \Big(m_2 T \iint\limits_T \rho_f^2\, dm_2\Big)^{1/2}. \tag{7}$$

If in inequality (7) we put $T = L' \cup L_n$, we obtain, by virtue of the relations (5) and (6),

$$\iint\limits_{L' \cup L_n} \rho_{f_n}\, dm_2 \leqslant \Big(3\varepsilon \iint\limits_{D(1)} \rho_{f_n}^2\, dm_2\Big)^{1/2} \leqslant \sqrt{3\varepsilon\pi n}, \quad n \in \mathbf{N}_3.$$

Choosing $T = E'$ in inequality (7) and applying relations (3) and (4), we obtain

$$\iint\limits_{E'} \rho_{f_n}\, dm_2 \leqslant (\pi^2 n \mu_n(E'))^{1/2} \leqslant \pi \sqrt{n\varepsilon}, \quad n \in \mathbf{N}_3. \tag{9}$$

We note now, by virtue of inequalities (6), that the image of the set $M' \setminus L_n$ under the action of the function f_n has a spherical area not exceeding $2\pi n \exp(-2n\delta)$ (taking multiplicity into account). Applying inequality (7) with $T = M' \setminus L_n$, we obtain

$$\iint\limits_{M' \setminus L_n} \rho_{f_n}\, dm_2 \leqslant (\pi \cdot 2\pi n \exp(-2n\delta))^{1/2} = o(1), \quad n \in \mathbf{N}_3.$$

Adding this relationship to inequalities (8) and (9), we obtain a contradiction with inequality (2).

2°. $v_n \to +\infty$. Then, for sufficiently large $n \in N_2$, we have $v_n(z) \geqslant 1$ for $z \in D(1) \setminus L_n$, where $m_2(L_n) \to 0$. We then reason as we did in 1°.

This completes the proof of the theorem.

Conjecture. Let $0 \leqslant u \leqslant 1$ be a subharmonic function in D(1). For arbitrary $\varepsilon > 0$ we have $\{z: u(z) < \varepsilon\} = L_\varepsilon \cup E_\varepsilon$, where $\mu(E_\varepsilon) \leqslant A_5 \varepsilon^\beta$, $m_2(L_\varepsilon) \leqslant A_5 \varepsilon^\beta$, with some absolute constant $\beta > 0$.

The proof of Theorem 1 shows that this conjecture would imply the inequality (1) with $\alpha < \beta/2$.

The authors wish to thank V. S. Azarin, S. U. Favorov, and A. L. Vol'berg for a useful discussion of this paper.

LITERATURE CITED

1. J. E. Littlewood, J. London Math. Soc., 27, No. 4, 387-392 (1952).
2. W. K. Hayman, J. d'Analyse Math., 36, 75-95 (1979).
3. B. Øksendal, Am. J. Math., 94, 331-342 (1972).
4. B. Øksendal, Pac. J. Math., 95, 179-192 (1981).

DYNAMICS OF THE CALOGERO–MOSER SYSTEM AND THE REDUCTION OF HYPERELLIPTIC INTEGRALS TO ELLIPTIC INTEGRALS

A. R. Its and V. Z. Énol'skii

UDC 517.946

We consider the algebraic curve C = (α, λ),

$$\lambda^3 - 3\lambda\wp(\alpha) - \wp'(\alpha) = 0, \quad \wp'^2 = 4\wp^3 - g_2\wp - g_3, \tag{1}$$

the three-sheeted covering torus $M = (\wp, \wp')$ [2], $\pi_M: C \to M$. The curve (1) represents one of the curves $\mathcal{K}_n$, introduced by Krichever [1]: $\det \| L - \lambda E \| = 0$, $E_{ij} = \delta_{ij}$, $L_{ij} = (\delta_{ij} - 1)\Phi_{ij} + \delta_{ij} y_i/2$, $\Phi_{ij} = \Phi(x_i - x_j;\ \alpha)$, $i, j = 1, \ldots, n$, $\Phi(x;\ \alpha) = \sigma(x - \alpha)\exp\{\zeta(\alpha)x\} / \sigma(x)\sigma(\alpha)$, whose coefficients $I_1, \ldots, I_n$ are the motion integrals of the Calogero–Moser system

$$H = \sum_{j=1}^{n} \frac{1}{2} y_j^2 - \sum_{i \neq j} \wp_{ij}, \quad \wp_{ij} = \wp(x_i - x_j), \quad n = \frac{g(g+1)}{2}, \quad g \in N. \tag{2}$$

if the quantities $I_j(x, y)$, $j = 1, \ldots, n$, are defined on the locus $\mathcal{L}_n$ [3],

$$\mathcal{L}_n = \{(y) \mid y_j = 0, j = 1, \ldots, n\} \times l_n, \quad l_n = \Big\{(x) \mid \sum_{j \neq i} \wp'_{ij} = 0, i = 1, \ldots, n\Big\}$$

[i.e., on the set of fixed points of (2)], then for n = 3 the curve $\mathcal{K}_n$ has the form (1).

LEMMA. The curve C is birationally equivalent to the curve $\hat{C} = (z, w)$,

$$w^2 = (z^2 - 3g_2)(z + 3e_1)(z + 3e_2)(z + 3e_3). \tag{3}$$

Proof. The curve C has genus g = 2 (the number of branchings of π_M is equal to two) and therefore it is hyperelliptic. In the neighborhoods of the points at infinity $P_j \in C$, j = 1, 2, 3 (situated over α = 0), the expansion of λ(α) has the form $\lambda = 1/\alpha \pm \alpha\sqrt{g/12} + O(\alpha^3)$, $\lambda = -2/\alpha + \alpha^3 g_2/36 + O(\alpha^5)$, respectively. Therefore, the meromorphic function of second order $z = (\lambda^2 - \wp(\alpha))/3$ establishes on C a canonical hyperelliptic structure (the point P_3 is a Weierstrass point). The asserted birational equivalence of the curves (1) and (3) follows from the equality

$$\wp = (z^3/27 + g_3)(z^2/3 - g_2)^{-1}, \tag{4}$$

which is proved by inserting z into (1). The equality (4) gives the covering $\pi_M: \hat{C} \to M$,

Leningrad State University. Institute of Theoretical Physics, Academy of Sciences of the USSR. Translated from Funktsional'nyi Analiz i Ego Prilozheniya, Vol. 20, No. 1, pp. 73-74, January-March, 1986. Original article submitted December 24, 1984.

 0016-2663/86/2001-0062$12.50