MEROMORPHIC FUNCTIONS OF FINITE ORDER WITH MAXIMAL DEFICIENCY SUM

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1. **Introduction.** In [1], Drasin proved the following theorem: Let f be a meromorphic function of finite order ρ with the property

$$\sum_{a\in C} \delta(a, f) = 2.$$

Then the following assertions are true:

1) 2ρ is a natural number ≥ 2;

2) $\delta(a,f) = p(a)/\rho$, $a \in \bar{C}$, where $p(a)$ is a nonnegative integer (this implies that the number of deficient values is no greater than 2ρ);

3) all deficient values are asymptotic.

This theorem proves Nevanlinna's conjecture of 1929. The proof of Drasin's theorem is extremely complex and, in addition to Nevanlinna's theory, uses a diverse collection of technical devices, such as Ahlfors' theory of covering surfaces, quasiconformal mappings, etc. In [2] one of the authors proposed a shorter proof based on Nevanlinna's theory and potential theory. In addition to 1)-3), the following assertions were proved:

4) $T(r, f) \sim r^\rho l_1(r)$, $r \to \infty$, where l_1 is a continuous function with the property that $l_1(2r) \sim l_1(r)$, $r \to \infty$;

5)

$$\sum_{\{a:\delta(a,f)>0\}} \log \frac{1}{|(f-a)(re^{i\theta})|} = \pi r^\rho l_1(r) |\cos \rho(\theta - l_2(r))| + o(r^\rho l_1(r)),$$

uniformly with respect to θ as $r \to \infty$, $re^{i\theta} \in C_0$. Here C_0 is the union of discs with centers at the points z_k and radii r_k such that

$$\sum_{|z_k|<R} r_k = o(R), \quad R \to \infty,$$

while l_2 is a continuous function such that $l_2(cr) - l_2(r) \to 0$ as $r \to \infty$ uniformly with respect to $c \in [1,2]$.

We will show that assertions 1)-5) can be proved with potential theory alone, and we will obtain a more general result.

In 1929 Nevanlinna conjectured that the deficiency relation

$$\sum_{a\in C} \delta(a, f) \leqslant 2$$

remains valid if the constants a are replaced by meromorphic functions $a(z)$ such that $T(r,a) = o(T(r,f))$. In this case $\delta(a,f)$ denotes $\delta(0,f - a)$. This hypothesis was recently proved by Osgood [3]. The proof was substantially simplified by Steinmetz [4]. It is natural to attempt to generalize Drasin's theorem to "small" meromorphic functions. The authors' attention was drawn to this problem during a visit of Yang Lo to Khar'kov in 1988. The following is valid:

THEOREM 1. Let f be a meromorphic function of finite lower order, and let S be a set of no more than a countable number of meromorphic functions a with the property

$$T(r, a) = o(T(r, f)), \quad r \to \infty. \tag{1.1}$$

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If

$$\sum_{a \in S} \delta(a, f) = 2, \tag{1.2}$$

then assertions 1)-5) are valid. Here 3) indicates that there exists a curve Γ extending to ∞ such that $f(z) - a(z) \to 0$ as $z \to \infty$, $z \in \Gamma$.

Theorem 1 was proved in [5] for entire functions.

We will not use the Osgood—Steinmetz theorem to prove Theorem 1. Our argument provides a new proof of this theorem for functions f of finite lower order.* An essentially new element is provided by Theorem 2 below, which replaces the fundamental theorem II and Osgood and Steinmetz' generalization.

Before we state Theorem 2, we will recall some notation from [2]. The difference of two subharmonic functions is called a δ-subharmonic function. In general, it is only defined quasieverywhere, i.e., inside some set of capacity zero. For a δ-harmonic function v we always set

$$v(z) = \lim_{\varepsilon \to 0} \frac{1}{2\pi} \int_0^{2\pi} v(z + \varepsilon e^{i\theta})\, d\theta$$

at all points z at which the limit exists. The natural order relation converts the linear space of δ-subharmonic functions into a lattice (the join $u \vee v = (u - v)^+ + v$ and meet $u \wedge v = u - (u - v)^+$ of δ-subharmonic functions are subharmonic functions). The Riesz charge is defined for every subharmonic function. There is also a natural order in the charge space: $\mu_1 \geq \mu_2$ if $\mu_1 - \mu_2$ is a (nonnegative) measure. The charge space with this order relation is a lattice: $\mu_1 \vee \mu_2 = (\mu_1 - \mu_2)^+ + \mu_2$, where μ^+ is the positive part in the Jordan decomposition of the charge μ.

THEOREM 2. Let $\{u_k\}_{k=1}^{\omega}$, $\omega \leq \infty$, be nonnegative δ-subharmonic functions such that

$$\sum_{k=1}^{\omega} u_k = \bigvee_{k=1}^{\omega} u_k. \tag{1.3}$$

We assume that the Riesz charge μ_k of the function u_k is uniformly bounded: $\mu_k \leq \mu$ for some measure μ and all k. Then

$$\sum_{k=1}^{\omega} \mu_k \leqslant 2 \bigvee_{k=1}^{\omega} \mu_k. \tag{1.4}$$

Our exposition will proceed according to the following plan. In Paragraph 2 we will prove Theorem 1, using Theorem 2 and the fundamental lemma of [2, Part II]. In Paragraph 3 we will prove Theorem 2.

2. **Proof of Theorem 1.** Without loss of generality, we can assume that $f(0) \neq \infty$ and

$$N(r, f) \sim T(r, f), \quad r \to \infty, \tag{2.1}$$

from which it follows that

$$m(r, f) = o(T(r, f)), \quad r \to \infty. \tag{2.2}$$

This means that ∞ is not an exceptional value in the sense of Valiron. We set $T(t) = T(r,f)$ and define the order ρ^* and the lower order ρ_* in the sense of Polya:

$$\rho^* = \sup\left\{p : \limsup_{r, B \to \infty} \frac{T(Br)}{B^p T(r)} = \infty\right\}; \tag{2.3}$$

$$\rho_* = \inf\left\{p : \liminf_{r, B \to \infty} \frac{T(Br)}{B^p T(r)} = 0\right\}. \tag{2.4}$$

*Our method can also be applied to arbitrary meromorphic functions. It makes it possible to obtain fundamental theorem II with small functions instead of constants, although in the case of infinite order it is necessary to use a slightly stronger smallness condition than (1.1). See [6].

We have the inequalities $\rho_* \leq \rho \leq \rho^*$. Henceforth we will use ρ to denote the lower order of the function f. Furthermore, we can show that it coincides with the order. It follows from the hypothesis of the theorem that $\rho_* < \infty$. For any $\lambda \in [\rho_*, \rho^*]$ there exists a sequence $r_j \to \infty$ of Polya peaks of order λ [7]. This means that for some sequence $\varepsilon_j \to 0$ we have

$$T(rr_j) \leqslant (1+\varepsilon_j) r^\lambda T(r_j), \quad \varepsilon_j \leqslant r \leqslant \varepsilon_j^{-1}. \tag{2.5}$$

We fix an arbitrary $\lambda \in [\rho_*, \rho^*]$ and a sequence r_j with the property (2.5).

Consider the δ-subharmonic functions

$$v_a = \log \frac{1}{|f-a|}, \quad a \in S$$

with Riesz charges ν_a. We define an operator A_j on the function v according to the rule $A_j v(z) = v(r_j z)/T(r_j)$, and on the charge ν according to the rule $A_j \nu(E) = \nu(r_j E)/T(r_j)$, $E \subset C$. It is clear that operators defined this way commute with the Laplace operator, i.e., the function $A_j v$ has charge $A_j \nu$. According to the Anderson—Baernstein theorem [8], conditions (2.5) imply that the family of δ-subharmonic functions $\{A_j v_a\}_{j=1}^{\infty}$ is relatively compact in the following sense. We can choose a subsequence of Polya peaks (which we again denote by r_j) so that

$$A_j v_a \to u_a, \tag{2.6}$$

$$A_j \nu_a \to \mu_a, \quad a \in S, \quad j \to \infty \tag{2.7}$$

where the u_a are δ-subharmonic functions with Riesz charges μ_a. Convergence in (2.6) occurs in L_{loc}^1, i.e., in mean with respect to area on each compactum, as well as in mean with respect to angular measure on each circle. The convergence of the charges in (2.7) is weak.

It follows from (2.2) that

$$u_a \geqslant 0, \quad a \in S. \tag{2.8}$$

We will show that (1.3) is satisfied. For any complex numbers x, a, and b we have

$$|x-a| \cdot |x-b| \geqslant \frac{|a-b|}{2} \min\{|x-a|, |x-b|\}.$$

It follows that for $a \neq b$, a, b $\in$ S, we have

$$v_a + v_b \leqslant v_a \vee v_b + \log|a-b|^{-1} + \log 2.$$

If we apply the operator A_j and permit j to go to ∞, we find, in view of (1.1), that $u_a + u_b \leq u_a \vee u_b$ when $a \neq b$. This means that at each point no more than one function u_a is different from zero. Thus, the functions u_a satisfy condition (1.3), i.e.,

$$\sum_{a \in S} u_a = \bigvee_{a \in S} u_a. \tag{2.9}$$

Note that

$$\frac{1}{2\pi} \int_0^{2\pi} u_a(re^{i\theta})\, d\theta = \lim_{j \to \infty} \frac{m(rr_j, 0, f-a)}{T(r_j)}. \tag{2.10}$$

It now follows from (2.5) that

$$\frac{1}{2\pi} \int_0^{2\pi} u_a(re^{i\theta})\, d\theta \leqslant r^\lambda, \quad 0 \leqslant r < \infty, \tag{2.11}$$

and, in particular, in virtue of (2.8),

$$u_a(0) = 0, \quad a \in S. \tag{2.12}$$

For any Borel σ-finite charge α we set $n(r,\alpha) = \alpha(\{z;\ |z| \leq r\})$,

$$N(r, \alpha) = \int_0^r n(t, \alpha) \frac{dt}{t},$$

if the integral converges absolutely. Let $\nu(E)$ be the number of poles of the function f with allowance for multiplicity on the set $E \subset \mathbf{C}$. Then ν is a (nonnegative) Borel σ-finite measure.

$$N(r, \nu) = N(r, f) \sim T(r), \quad r \to \infty. \tag{2.13}$$

In virtue of (2.5),

$$N(rr_j, \nu) \leqslant 2r^{\lambda} T(r_j), \quad \varepsilon_j \leqslant r \leqslant \varepsilon_j^{-1}, \tag{2.14}$$

which implies

$$n(rr_j, \nu) \leqslant 2e^{\lambda} r^{\lambda} T(r_j), \quad \varepsilon_j/e \leqslant r \leqslant (e\varepsilon_j)^{-1}. \tag{2.15}$$

Condition (2.15) means that the family of measures $\{A_j\mu\}$ is uniformly bounded on compacta, so, choosing a subsequence of Polya peaks if necessary, we can assume that

$$A_j\nu \to \mu \geqslant 0. \tag{2.16}$$

It also follows from (2.15) that

$$n(r, \mu) \leqslant 2e^{\lambda} r^{\lambda}, \quad 0 \leqslant r < \infty. \tag{2.17}$$

We will show that

$$N(r, A_j\nu) \to N(r, \mu), \quad j \to \infty, \quad 0 \leqslant r < \infty. \tag{2.18}$$

Let $\varepsilon > 0$. We have

$$|N(r, A_j\nu) - N(r, \mu)| \leqslant \int_0^{\varepsilon} \frac{n(t, \mu)}{t} dt + \int_0^{\varepsilon} \frac{n(t, A_j\nu)}{t} dt + \left| \int_{\varepsilon}^{r} (n(t, A_j\nu) - n(t, \mu)) \frac{dt}{t} \right|.$$

The first term is no greater than $2e^{\lambda}\lambda^{-1}\varepsilon^{\lambda}$ in virtue of (2.17), and the second term is equal to

$$N(\varepsilon, A_j\nu) = N(r_j\varepsilon, \nu)/T(r_j) \leqslant 2\varepsilon^{\lambda}$$

in virtue of (2.14); also, the third term approaches zero as $j \to \infty$ in virtue of the weak convergence of (2.16). This proves (2.18).

Now, let $\kappa_a(E)$ be the number of poles of $a \in S$ (with allowance for multiplicity) on the set $E \subset \mathbf{C}$. It is clear that

$$\nu_a \leqslant \nu + \varkappa_a, \quad a \in S. \tag{2.19}$$

On the other hand, it follows from (1.1) and (2.5) that $A_j\kappa_a \to 0$ as $j \to \infty$. If we apply the operator A_j to inequality (2.19) and pass to the limit as $j \to \infty$, we find, in view of (2.7) and (2.16), that $\mu_a \leq \mu$. Since (2.9) is satisfied, application of Theorem 2 yields

$$\sum_{a \in S} \mu_a \leqslant 2 \bigvee_{a \in S} \mu_a \leqslant 2\mu. \tag{2.20}$$

We now use condition (1.2). We fix an arbitrarilly small $\varepsilon > 0$ and choose a finite subset $S' \subset S$ such that

$$\sum_{a \in S'} \delta(a, f) \geqslant 2 - \varepsilon.$$

In view of (2.13) we have

$$\sum_{a \in S'} m(rr_j, 0, f - a) \geqslant (2 - 2\varepsilon) T(rr_j) \geqslant (2 - 3\varepsilon) N(rr_j, \nu)$$

for fixed r and $j \to \infty$. Dividing by $T(r_j)$ and passing to the limit, in view of (2.10) and (2.18), we find that

$$\sum_{a \in S'} \frac{1}{2\pi} \int_0^{2\pi} u_a(re^{i\theta}) d\theta \geqslant (2 - 3\varepsilon) N(r, \mu).$$

On the other hand, Jensen's formula and (2.12) yield

$$\frac{1}{2\pi}\int_0^{2\pi} u_a(re^{i\theta})\,d\theta = N(r,\ \mu_a).$$

Thus,

$$\sum_{a\in S} N(r,\ \mu_a) \geqslant \sum_{a\in S'} N(r,\ \mu_a) \geqslant (2-3\varepsilon)\,N(r,\ \mu).$$

If we permit ε to approach zero,

$$\sum_{a\in S} N(r,\ \mu_a) \geqslant 2N(r,\ \mu).$$

Together with (2.20), this yields

$$\sum_{a\in S} \mu_a = 2\bigvee_{a\in S} \mu_a = 2\mu. \tag{2.21}$$

We will now show that all of the functions u_a are subharmonic and continuous. First of all, the function $w = \sum_{a\in S} u_a$ is subharmonic, since, in view of (2.21), its Riesz charge is $2\mu \geq 0$. Moreover, it follows from (2.21) that $\mu \geq \mu_a$ for $a \in S$, so the function $w_a = w - 2u_a$ is subharmonic. It is easy to see that $u_a(z) > 0$ if and only if $w_a(z) < 0$. Since w_a is upper semicontinuous, the set $D_a = \{z\colon u_a(z) > 0\}$ is open. Application of the maximum principle to the function w_a shows that all of the connected components of D_a are singly connected. On this set D_a we have $u_b(z) \equiv 0$ for all $b \neq a$, so $\mu_b \mid_{D_a} = 0$, and it then follows from (2.21) that $\mu_a \mid_{D_a} = 0$. Thus, the function $u_a \geq 0$ is harmonic in D_a and equal to zero outside D_a. It follows that it is subharmonic and continuous.

It now follows from the subharmonic form of the D'Anjou—Karleman—Ahlfors theorem [9] that the set S is finite (and card S $\leq 2\lambda$). Moreover, the total number of connected components of the sets D_a is also finite. We can now use the following Fundamental Lemma, which was proved in [2, Part II].

LEMMA 1. Let $\{D_a\}$ be a set of pairwise disjoint open sets consisting of a finite number of singly connected regions, and let $u_a \not\equiv 0$ be a nonnegative subharmonic function with carrier in D_a. Assume that the Riesz measure μ_a of these functions satisfies (2.21) and

$$\sum_{a\in S}\int_0^{2\pi} u_a(re^{i\theta})\,d\theta = \begin{cases} O(r^{\lambda+\varepsilon}), & r\to\infty;\\ O(r^{\lambda-\varepsilon}), & r\to 0,\end{cases} \tag{2.22}$$

where $0 < \varepsilon < 1/4$, card S $< \infty$. Then there exist a natural number $n \geq 2$, $|\, n/2 - \lambda \,| < 1/2$ and a number $\theta_0 \in [,0,2\pi]$ such that

$$w(re^{i\theta}) \stackrel{\mathrm{def}}{=} \sum_{a\in S} u_a(re^{i\theta}) = cr^{n/2}\left|\cos\frac{n}{2}(\theta-\theta_0)\right|. \tag{2.23}$$

The functions u_a satisfy the conditions of Lemma 1 ((2.22) is satisfied when $\varepsilon = 0$ because of (2.11)). Thus, (2.23) holds. We will now determine the constant c. Toward this end, we set $r = 1$ in (2.23) and integrate with respect to θ:

$$c = \frac{1}{4}\int_0^{2\pi} w(e^{i\theta})\,d\theta.$$

We now use Jensen's formula, (2.21), (2.18), and (2.13), which yields

$$\frac{1}{2\pi}\int_0^{2\pi} w(e^{i\theta})\,d\theta = \frac{1}{2\pi}\sum_{a\in S}\int_0^{2\pi} u_a(e^{i\theta})\,d\theta = \sum_{a\in S} N(1,\ \mu_a) =$$
$$= 2N(1,\ \mu) = 2\lim_{j\to\infty} N(1,\ A_j\nu) = 2\lim_{j\to\infty}\frac{N(r_j,\ \nu)}{T(r_j)} = 2,$$

so that $c = \pi$.

We can summarize as follows: Let W_n be the set of all subharmonic functions of the form

$$w(re^{i\theta}, \theta_0) = \pi r^{n/2}\left|\cos\frac{n}{2}(\theta-\theta_0)\right|, \quad 0 \leq \theta_0 < 2\pi.$$

We have proved the following

Proposition 1. Let f be a meromorphic function satisfying condition (1.2), and let r_j be a sequence with property (2.5). Then the set S is finite and for some subsequence of indices j we have (2.6), and

$$w = \sum_{a \in S} u_a \in W_n.$$

It follows from comparison of (2.23) and (2.11) that $2\lambda = n \in \mathbf{N}$. Since the possible orders λ of the Polya peaks fill the segment $[\rho_*, \rho^*]$, this interval must degenerate to a point, i.e., $\rho_* = \rho^* = \rho = n/2$. It follows from the definitions of ρ_* and ρ^* ((2.3) and (2.4)) that for any $\varepsilon > 0$ there exist $r_0 > 1$ and $t_0 > 1$ such that

$$T(tr) \leq t^{\rho+\varepsilon} T(r), \quad t > t_0, \quad r > r_0; \tag{2.24}$$

$$T(tr) \leq t^{\rho-\varepsilon} T(r), \quad t < t_0^{-1}, \quad tr > r_0. \tag{2.25}$$

These properties make it possible to replace (2.5) in the proof of Proposition 1.

Proposition 2. Let f be a meromorphic function satisfying conditions (1.2), (2.24), and (2.25). For any sequence $r_j \to \infty$ we define operators A_j as at the beginning of the proof. Then $A_j[\log|f-a|^{-1}] \to u_a$, where $w = \sum u_a \in W_n$.

The proof of Proposition 2 is the same as that of Proposition 1, with the following changes. Application of the Anderson—Baernstein theorem for the proof of (2.6) and (2.7) makes it possible to substitute conditions (2.24) and (2.25) for (2.5). Instead of (2.11), we obtain

$$\int_0^{2\pi} u_a(re^{i\theta})\, d\theta \leq \begin{cases} r^{\rho+\varepsilon}, & r > t_0 \\ r^{\rho-\varepsilon}, & r < t_0^{-1} \end{cases} \tag{2.26}$$

from (2.24) and (2.25). The argument proving (2.16) and (2.18) requires some obvious changes. For example, the role of (2.17) is now played by the inequality

$$n(r, \mu) \leq \begin{cases} 2e^{\rho-\varepsilon} r^{\rho-\varepsilon}, & r < (t_0 e)^{-1}; \\ 2e^{\rho+\varepsilon} r^{\rho+\varepsilon}, & r > t_0. \end{cases}$$

Finally, when Lemma 1 is applied, (2.26) is used instead of (2.11).

In view of (2.13), Proposition 2 implies that

$$T(cr)/T(r) \sim N(cr, f)/N(r, f) \to c^{\rho}$$

as $r \to \infty$ uniformly with respect to $c \in [1,2]$. Setting $T(r) = r^{\rho} l_1(r)$, we obtain $l_1(cr) \sim l_1(r)$, which proves assertion 4) of Theorem 1. Assertion 1) was proved above. We will prove assertion 5), from which assertions 2) and 3) follow.

Note that L^1_{loc} is a metrizable space. Let dist denote some metric in this space. The set W_n is compact in L^1_{loc}. We set

$$v_t(z) = \frac{1}{t^{\rho} l_1(t)} \sum_{a \in S} \log\frac{1}{|(f-a)(tz)|}$$

and we will show that

$$\operatorname{dist}(v_t, W_n) \to 0, \quad t \to \infty. \tag{2.27}$$

Assume that (2.7) is not satisfied. Then there exists a sequence $t_j \to \infty$ such that $\operatorname{dist}(v_{t_j}, W_n) \geq \varepsilon > 0$. Taking this sequence for r_j, we now apply Proposition 2. We find that for some subsequence $v_{t_j} \to w$, where $w \in W_n$, which is a contradiction. Relation (2.27) is therefore proved.

Let $w^t \in W_n$ be the element closest to v_t. We will show that

$$\operatorname{dist}(w^t, w^{ct}) \to 0, \quad t \to \infty, \tag{2.28}$$

uniformly with respect to $c \in [1,2]$. Assume that this is not so. Then

$$\operatorname{dist}(w^{t_j}, w^{c_j t_j}) \geqslant \varepsilon > 0 \tag{2.29}$$

for some sequences $t_j \to \infty$ and $c_j \in [1,2]$. We have

$$w^{c_j t_j}(z) = v_{c_j t_j}(z) + o(1) = c_j^{-\rho} v_{t_j}(c_j z) + o(1) = \\ = c_j^{-\rho} w^{t_j}(c_j z) + o(1) = w^{t_j}(z) + o(1),$$

since $c^{-\rho}w(cz) = w(z)$ for any $w \in W$ and $c > 0$. We have obtained a contradiction with (2.29), which proves (2.28).

If $w^t = w(\cdot,\varphi_0(t))$, it follows from (2.28) that

$$|\exp(i\varphi_0(ct)) - \exp(i\varphi_0(t))| \to 0, \ t \to \infty,$$

uniformly with respect to $c \in [1,2]$. In view of (2.27), we can choose a continuous function $\theta_0(t)$ so that

$$v_t = w(.,\ \theta_0(t)) + o(1),\ \theta_0(ct) = \theta_0(t) + o(1),\ t \to \infty.$$

This, together with V. S. Azarin's theorem on convergence of subharmonic functions with respect to 1-measure [10], yields assertion 5).

3. **Proof of Theorem 2.** The proof is basically elementary. Condition (1.3) means that the functions u_k have disjoint support, i.e., at each point no more than one of these functions is nonzero. It follows that the charges μ_k have Borel carriers that intersects no more than twice, from which (1.4) immediately follows.

Carrying out this plan, however, is associated with certain technical difficulties resulting from the possible discontinuity of the functions u_k and the complex mechanism of the sets $\{z: u_k(z) > 0\}$.

We should note that it is sufficient to prove Theorem 2 for finite ω. Indeed, (1.3) implies the same property for any finite set of indices k. If, however, (1.4) is proved for any finite set of indices, then (1.4) yields a limit, when we take into account that $\mu_k \leq \mu$. We will thus assume that $q = \omega < \infty$.

An equivalent form of Theorem 2 that is of independent interest is

THEOREM 2′. Let $w_1,\ldots,w_q$ be subharmonic functions, $q \geq 2$, and let w be their join. We assume that for any $i \neq j$ we have

$$w = w_i \vee w_j. \tag{3.1}$$

Then the function

$$h = w + \bigwedge_{k=1}^{q} w_k = \sum_{k=1}^{q} w_k - (q-2)w \tag{3.2}$$

is subharmonic.

To derive Theorem 2′ from Theorem 2 we set $u_k = w - w_k$ and denote the Riesz measure of function w by μ. Since functions w_k are subharmonic, we have $\mu \geq \mu_k$, and, by (1.4), $\Sigma\,\mu_k \leqslant 2\mu$, i.e., $2w - \sum u_k = (q-2)w - \sum w_k = h$ is subharmonic.

Now we will derive Theorem 2 from Theorem 2′. We set $\mu = \vee\,\mu_k$, $\nu_k = \mu - \mu_k \geq 0$. Let w be a δ-subharmonic function with Riesz charge μ. We set $w_k = w - u_k$. The functions w_k are subharmonic, since their Riesz charges are ν_k. Now, (3.1) follows from (1.3). In particular, w is a subharmonic function, since it is the join of subharmonic functions. Finally, (3.2) implies

$$0 \leqslant \sum \nu_k - (q-2)\mu = \sum(\mu - \mu_k) - (q-2)\mu = 2\mu - \sum \mu_k,$$

which is the same as (1.4).

The statement of Theorem 2′ has an important advantage. It makes it possible to deal with continuous functions only.

For any subharmonic function v we set $v^\varepsilon(z) = \max\{v(\zeta) : |\zeta - z| \leq \varepsilon\}$. It is easy to see that the function v^ε is always continuous and subharmonic. In addition, the operation $v \to v^\varepsilon$ commutes with taking joins. Assume that Theorem 2′ is proved for continuous subharmonic functions. Let $w_1,\ldots,w_q$ be arbitrary subharmonic functions with property (3.1). Then

$w^\varepsilon = w_i^\varepsilon \vee w_j^\varepsilon$ for $i \neq j$, and we find that the function $h_\varepsilon = w^\varepsilon + \bigwedge_{k=1}^{q} w_k^\varepsilon$ is subharmonic. If $\varepsilon \to 0$, then $h_\varepsilon \to h$, monotonically decreasing. Thus, h is a subharmonic function. This property was pointed out to the authors by V. S. Azarin. Thus, it is sufficient to prove Theorem 2 for continuous functions.

We now introduce some notation. Let D be a bounded open set. We say that a point $z_0 \in \mathbf{C} \setminus D$ is reachable from D if there exists a curve $\gamma(t)$: $0 \leq t \leq 1$ such that $\gamma(t) \in D$, $0 < t < 1$ and $\gamma(1) = z_0$. The set of points that are reachable from D is a Borel set [11]. We define the function $G_D(z,\zeta)$ as follows: if z and ζ are elements of the same connected component of the set D, then $G_D(z,\zeta)$ is the Green's function of this component with pole at ζ; otherwise $G_D(z,\zeta) = 0$. The function $z \to G_D(z,\zeta)$ defined in this way is subharmonic in $\mathbf{C} \setminus \{\zeta\}$. Its Riesz measure $\omega_D(\zeta,\cdot)$ is called the harmonic measure with respect to D at the point ζ.

LEMMA 2. Let E^* is the set of points not reachable from D. Then $\omega_D(\zeta,E^*) = 0$, $\zeta \in D$.

This is a known result (see, for example, [12]). The clearest proof is based on a probabilistic interpretation of the harmonic measure: $\omega_D(\zeta,E)$ is the probability that a Brownian particle leaving the point ζ first leaves D through the set E.

LEMMA 3. Let v be continuous δ-subharmonic function, $D = \{z: v(z) \neq 0\}$, and let E^* be the set of points that cannot be reached from D. If ν is the Riesz charge of the function v, then its restriction to the set E^* is equal to zero: $\nu \mid_{E^*} = 0$.

Proof. It is sufficient to prove the lemma for a finite function v, so that for any $R > 0$ and any δ-subharmonic function v there exists a finite δ-subharmonic function v_R such that $v(z) = v_R(z)$, $|z| < R$.

We can assume that D is a domain. Indeed, if $\{D_j\}$ is the set of all connected components of D, we set

$$v_j(z) = \begin{cases} v(z), & z \in D_j \\ 0, & z \notin D_j. \end{cases}$$

Having proved the lemma for the functions v_j, we can now prove it for the general case.

Thus, let v be a finite continuous δ-subharmonic function and let $D = \{z: v(z) \neq 0\}$ be a domain. Then v is the Green potential

$$v(z) = -\int_D G_D(z, \zeta)\, d\nu_\zeta, \quad z \in \mathbf{C}. \tag{3.3}$$

(Representation (3.3) holds everywhere in $\mathbf{C}$ in virtue of our stipulation concerning Green's function.) Moreover,

$$G_D(z, \zeta) = \int_C \log|z-t|\,\omega_D(\zeta, dt) - \log|z-\zeta|.$$

Substituting this expression into (3.3) and applying the Frobenius theorem, we obtain

$$v(z) = \int_D \log|z-\zeta|\,d\nu_\zeta - \int_C \log|z-\zeta|\,d\alpha_\zeta, \tag{3.4}$$

where the charge α is defined by the expression

$$\alpha(E) = \int_D \omega_D(\zeta, E)\,d\nu_\zeta, \quad E \subset \mathbf{C}.$$

In particular, $\alpha(E) = 0$ for any $E \subset E^*$ in virtue of Lemma 2. Now, it follows from representation (3.4) that the restriction of the Riesz charge of the function v to E^* is equal to zero. The lemma is proved.

LEMMA 4. Let $\{D_j\}_{j=1}^{q}$ be a set of pairwise disjoint open sets in $\mathbf{C}$. Then the set of points reachable simultaneously from three different D_j is no larger than countable.

Proof. It is sufficient to prove the following: if B_1, B_2, and B_3 are pairwise disjoint domains, the set of points reachable simultaneously from all three domains B_j consists of no more than two points. Assume that this is not so. Then there exist three different points z_1, z_2, and z_3 that are reachable from B_1, B_2, and B_3. We choose points $w_i \in B_i$, $i \leq i \leq 3$. Each point w_i can be connected by three disjoint curves belonging to B_i to the points z_1, z_2, z_3. The union of all of these curves and the points w_i and z_i forms a graph $K_{3,3}$ that can be imbedded in the plane, which, as we know, is impossible. The contradiction proves the lemma.

Now we can finish the proof of Theorem 2. Let $D_k = \{z: u_k(z) > 0\}$, and let D_k^* be the union of D_k and the set of all points reachable from D_k. Then, by Lemma 3, the charge μ_k is concentrated on D_k^*, i.e., $\mu_k(E) = 0$ for any $E \subset C \backslash D_k^*$. By Lemma 4, the set X of points contained in three or more D_k^* is no larger than countable. Since u_k is continuous, $\mu_k(E) = 0$ for all $E \subset X$. Thus, the charges μ_k have Borel carriers $D_k^* \backslash X$ that intersect no more than twice. This, together with the inequality $\nu_1 + \nu_2 \leq 2(\nu_1 \vee \nu_2)$, which is valid for all charges, implies (1.4). The theorem is proved.

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