where

$$b(\lambda)=\frac{1}{2i\lambda}W\{e^-(\lambda,x),\ e^+(\lambda,x)\}=\frac{1}{2i\lambda}e^+(\lambda,0)e^-(\lambda,0)\{m^-(\lambda^2)-m^+(\lambda^2)\},$$

it follows that the equality $r^{\pm}(\lambda) = 0$ for $|\lambda|\geqslant C$ is equivalent to the condition $m^+(\lambda^2)=m^-(\lambda^2)$ for $\lambda^2\geqslant C^2$.

Since Eq. (1) can be rewritten in the form $-y''+[q(x)-C^2]y=(\lambda^2-C^2)y$, it follows that $m^{\pm}_{q-C^2}(\lambda^2-C^2)=m^{\pm}_q(\lambda^2)$.

From here it follows that if $m^+_q(\lambda^2)=m^-_q(\lambda^2)$ for $\lambda^2\geqslant C^2$, then $m^+_{q-C^2}(\lambda^2-C^2)=m^-_{q-C^2}(\lambda^2-C^2)$ for $\lambda^2-C^2\geqslant 0$.

Setting $\lambda_1^2=\lambda^2-C^2$, we note that the function

$$n(\lambda_1)=\begin{cases} m^+_{q-C^2}(\lambda_1^2)\ \mathrm{Im}\,\lambda_1>0;\\ m^-_{q-C^2}(\lambda_1^2)\ \mathrm{Im}\,\lambda_1<0\end{cases}$$

is analytic with respect to λ_1 everywhere outside some interval of the imaginary axis. Consequently, by Theorem 2, $q(x)-C^2=q_0(x)\in\overline{B(\mu)}$.

Thus, we have proved the following fact: if the reflection coefficients are finite: $r^+(\lambda)=r^-(\lambda)=0$ for $|\lambda|\geqslant C$, then $q(x)=C^2+q_0(x)$, where $q_0(x)\in\overline{B(\mu)}$.

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A NEW PROOF OF DRASIN'S THEOREM ON MEROMORPHIC FUNCTIONS OF FINITE ORDER WITH MAXIMAL DEFICIENCY SUM. II

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This paper is the continuation of [1]. We start with some remarks.

First of all, it is sufficient to prove the Fundamental Lemma under the assumption that D_j are domains. Indeed, let $D_j=\bigcup_k D_{jk}$ be the decomposition into connected components. We set

$$u_j=\sum_k u_{jk},\ \mathrm{supp}\,u_{jk}\subset D_{jk},$$

$\mu_j=\sum_k\mu_{jk}$, μ_{jk} is the Riesz measure of the function μ_{jk}. Then (6.5) yields $\sum_{j,k}\mu_{jk}\geqslant 2\sum_k\mu_{ik}\geqslant 2\mu_{in}$ for all i and n. The remaining conditions of the lemma are maintained.

Thus, we assume in the sequel that D_j are connected and we number D_j, u_j, μ_j by the same index $1\leqslant j\leqslant n$.

From the maximum principle for subharmonic functions there follows that the domains D_j are unbounded.

The set of all Borel measures in $\boldsymbol{C}$ is partially ordered by the relation $\leq$. For a finite family of measures $\nu_1,\dots,\nu_k$ there exists the least upper bound $\nu_1\vee\nu_2\vee\cdots\vee\nu_k$. Condition (6.5) can be rewritten in the form

$$\sum_j\mu_j\geqslant 2(\mu_1\vee\cdots\vee\mu_n). \tag{8.1}$$

Let ν be a Borel measure in $\boldsymbol{C}$ and let A be a Borel set. The restriction of the measure ν to A is defined in the following manner: $\nu|_A(E)=\nu(A\cap E)$ for each Borel set $A\subset\boldsymbol{C}$.

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Everywhere in the sequel, the closure and the boundary are considered in $\bar{C}$.

8. Proof of the Fundamental Lemma in a Special Case. In order to make clear the idea of the proof, first we assume that the domains $D_j \subset \bar{C}$ are Jordan domains. Let μ_{jj} be the restriction of the measure μ_j to the set $\bar{D}_j \setminus \bigcup_{i \neq j} \partial D_i$, $1 \leqslant j \leqslant n$, and

$$\mu_{jk} = \mu_j |_{\partial D_j \cap \partial D_k \setminus \{\infty\}}, \ k \neq j, \ 1 \leqslant j, \ k \leqslant n.$$

We note that the set of points, belonging at once to three distinct $\bar{D}_j$, is finite (see Lemma 6 below). (The simplifying assumption about the Jordan property of the domains has been formulated first of all for this.) In addition $\mu_j(E) = 0$ for any finite set $E \subset C$ since $u_j \geqslant 0$. Consequently,

$$\mu_j = \sum_{k=1}^{n} \mu_{jk}, \ 1 \leqslant j \leqslant n. \tag{8.2}$$

If the unordered pairs $\{i, k\}$ and $\{j, p\}$ do not coincide, then the Borel supports of the measures μ_{ik} and μ_{jp} can be selected to be disjoint. Taking into account this circumstance, as well as (8.1) and (8.2), we obtain

$$\sum_{i,j=1}^{n} \mu_{ij} = \sum_{i=1}^{n} \mu_i \geqslant 2 \Big(\sum_{j=1}^{n} \mu_{jj} + \sum_{\substack{i,j \\ i<j}} (\mu_{ji} \vee \mu_{ij}) \Big)$$

or

$$\sum_{\substack{i,j \\ i<j}} (\mu_{ij} + \mu_{ji}) \geqslant \sum_{j=1}^{n} \mu_{jj} + 2 \sum_{\substack{i,j \\ i<j}} (\mu_{ij} \vee \mu_{ji}). \tag{8.3}$$

We always have $\mu_{ij} + \mu_{ji} \leqslant 2(\mu_{ij} \vee \mu_{ji})$ and equality is possible only if $\mu_{ij} = \mu_{ji}$. Therefore, from (8.3) there follows

$$\mu_{jj} = 0, \ 1 \leqslant j \leqslant n \tag{8.4}$$

$$\mu_{ij} = \mu_{ji}, \ 1 \leqslant j < i \leqslant n \tag{8.5}$$

From (8.4) there follows that the functions u_j are harmonic in D_j. Further, it is easy to see that the closed support of the measure μ_j coincides with $\partial D_j \setminus \{\infty\}$. Therefore, from (8.4) there follows that the entire boundary of the domain D_j is covered by the boundaries of the other domains D_i. Here we use again the assumption that the domains D_j are Jordan domains. Thus, $\bigcup_j \bar{D}_j = \bar{C}$. The sphere is partitioned by curves, homeomorphic to a segment, into n simply connected domains D_j. The curves are called edges and their extremities are called vertices. A vertex is said to be odd if an odd number of edges converge to it. The set of odd vertices will be denoted by Q. The number of odd vertices is even. Indeed, if we sum the number of edges, converging to each vertex, then we obtain twice the total number of edges, i.e., an even number. Consequently, in this sum the number of odd terms is even.

By a curve on a Riemann surface F we mean a continuous mapping $[0, 1] \to F$. Let $\Gamma \subset C$ be a curve that does not pass through the vertices and it is transversally intersecting the edges. The number of the intersections with the edges is denoted by $n(\Gamma)$. The number of rotations of a closed curve Γ with respect to a point $z \in C$ is denoted by $\mathrm{ind}_z \Gamma$.

LEMMA 2. For closed curves Γ we have

$$n(\Gamma) \equiv \sum_{z \in Q} \mathrm{ind}_z \Gamma \pmod 2.$$

Proof. By a small deformation we achieve that the curve Γ should have a finite number of self-intersections. We shall deform continuously the curve Γ, contracting it into the point $z_0 \in D_1$. The deformation is carried out in such a manner that the intermediate curves should have a finite number of self-intersections, a finite number of transversal intersections with the interior points of the edges, and the points of self-intersection of the curves should not be at the vertices. When during the deformation process the curve passes through the vertex z so that the number $\mathrm{ind}_z \Gamma$ varies by 1, the number $n(\Gamma)$ obtains an even increment if the vertex z is even, and an odd increment if z is an odd vertex. This proves the lemma.

We consider a two-sheeted covering of the sphere $\bar{C}$ by some Riemann surface F, ramified exactly over Q. Such a covering $\pi\colon F \to \bar{C}$ exists since Q contains an even number of points.

The covering π is unramified over the simply connected domains D_j and, therefore, each of them has exactly two preimages: $\tilde{D}_j$ and $\tilde{D}_{j+n}$. Further we assume that all the considered curves intersect transversally the edges and do not pass through the vertices. Lifting the vertices to the surface F, we define $n(\Gamma)$ for the curves Γ on F in the same way as above. Clearly,

$$n(\Gamma)=n(\pi(\Gamma)). \tag{8.6}$$

The closed curve $\Gamma\subset \mathbf{C}$ is lifted to a closed curve on F if and only if it goes around the branch points an even number of times, i.e., if

$$\sum_{z\in Q}\operatorname{ind}_z\Gamma\equiv 0 \pmod 2.$$

From here, taking into account (8.6) and Lemma 2, there follows that $n(\Gamma)\equiv 0 \pmod 2$ for all closed curves Γ on F.

Now the domains $\tilde{D}_j$, $1\leqslant j\leqslant 2n$, can be divided into two classes in the following manner: $\tilde{D}_j$ belongs to the k-th class ($k = 0, 1$) if $n(\Gamma)\equiv k \pmod 2$ for any curve Γ with origin in $\tilde{D}_1$ and endpoint in $\tilde{D}_j$. This definition is correct since for closed curves on F we have $n(\Gamma)\equiv 0 \pmod 2$. Now we can lift the function u_j to F by setting

$$\tilde{u}_j(z)=u_j\circ\pi(z),\ z\in\tilde{D}_j;\ \tilde{u}_j(z)=0,\ z\in F\setminus\tilde{D}_j;$$
$$\tilde{u}_{j+n}(z)=u_j\circ\pi(z),\ z\in\tilde{D}_{j+n};\ \tilde{u}_{j+n}(z)=0,\ z\in F\setminus\tilde{D}_{j+n},$$
$$1\leqslant j\leqslant n.$$

On F we consider the function

$$v=\sum_{j=1}^{2n}(-1)^{k(j)}\tilde{u}_j,$$

where $k(j)$ is the class of the domain $\tilde{D}_j$. The Riesz charge of this δ-subharmonic function is equal to

$$\sum_{j=1}^{2n}(-1)^{k(j)}\tilde{\mu}_j=\sum_{j,i}(-1)^{k(j)}\tilde{\mu}_{ji}. \tag{8.7}$$

Here $\tilde{\mu}_j$ is the Riesz measure of the function $\tilde{\mu}_j$ on F, while $\tilde{\mu}_{ji}$ is the restriction of $\tilde{\mu}_j$ to $\partial\tilde{D}_j\cap\partial\tilde{D}_i\setminus\{\infty\}$. If $\tilde{\mu}_{ji}\neq 0$, then the domains $\tilde{D}_j$ and $\tilde{D}_i$ have a common edge on the boundaries and, consequently, belong to distinct classes. From (8.5) there follows that $\mu_{ij}=\mu_{ji}$ and, therefore, the expression (8.7) is equal to 0 and the function v is harmonic in $F\setminus\pi^{-1}(\{\infty\})$.

Now we make use of the condition (6.7), from where it follows that

$$v(z)=O(|\pi(z)|)^{\lambda+\varepsilon},\ \pi(z)\to\infty, \tag{8.8}$$
$$v(z)=O(|\pi(z)|)^{\lambda-\varepsilon},\ \pi(z)\to 0. \tag{8.9}$$

Let h be a multivalued analytic function on F such that $v = \operatorname{Re} h$ (the various branches of h differ by constant, purely imaginary terms). The derivative $y = dh/d\pi$ is a single-valued function on $F\setminus\pi^{-1}(\infty)$. From (8.8) there follows that this function is meromorphic on F. At each vertex on F there converge an even number >2 of domains $\tilde{D}_j$. Consequently, when the point $z\in F$ goes around a vertex, the function v changes sign at least four times. From here it follows that the vertices, lying above a finite part of the plane, are zeros of the differential dh and, moreover, the branch points of F over $\mathbf{C}$ are necessarily multiple zeros. The differential $d\pi$ has simple zeros at the branch points. Thus, the meromorphic function y has zeros at all the vertices and can have poles only above the point ∞. From (8.8) there follows that the total order of poles does not exceed $2(\lambda - 1 + \varepsilon)$. Further, from (8.9) there follows that the total order of zeros, projecting into 0, is not less than $2(\lambda - 1 - \varepsilon)$. There exists a unique integer $k\geqslant 0$, satisfying the inequalities $2(\lambda-1-\varepsilon)\leqslant k\leqslant 2(\lambda-1+\varepsilon)$. Consequently, the function y does not have other zeros besides those projecting into 0. In particular, there are no vertices besides 0 and ∞. The function v has the form $v(z)=\operatorname{Re}(az^{n/2})$, where $n=k+2$, $a\in\mathbf{C}$, $z\in\mathbf{C}$. Clearly, in this formula n is exactly the number of domains D_j. The fundamental lemma has been proved under the a priori assumption that the D_j's are Jordan domains.

9. Unattainable Boundary Points. The difficulty of the proof of the Fundamental Lemma in the general case is connected with the existence of unattainable boundary points on the

boundaries of the domains D_j, which cannot be excluded a priori. In the general case the proof will be the same as in Sec. 9, but some of its steps require additional justification.

Let $D \subset \overline{C}$ be a domain. A point $z_0 \in \partial D$ is said to be attainable (from D) if there exists a curve $\Gamma \subset D$, ending at the point z_0. The set of attainable boundary points (a.b.p.) is a Borel set [2].

Assume that the domain D possesses a Green function. We fix a point $z_0 \in D$ and we consider the Green function with pole at z_0. We define it to be equal to zero outside D and we denote the extended function by g. The function g is subharmonic in $\overline{C} \setminus \{z_0\}$ and continuous if the domain D is regular for the Dirichlet problem. The Riesz measure of the function g is concentrated on ∂D and is nothing else but the harmonic measure at the point z_0 with respect to D [3, Chap. IX, Sec. 4]. For various choices of z_0 the corresponding harmonic measures are mutually absolutely continuous.

LEMMA 3. Assume that the boundary of the domain D is regular.* Then the harmonic measure of the set of unattainable points of ∂D is equal to 0.

Proof. Assume that the lemma is false. Then there exists a closed subset K of the set of unattainable boundary points, whose harmonic measure is >0.

Let v(z) be the infimum of the functions w(z), harmonic in D, continuous in $\overline{D}$, and satisfying the conditions $w(z) \geqslant 0$ in $\overline{D}$ and $w(z) \geqslant 1$ on K. The function v is harmonic and bounded in D. By assumption, $v(z) > 0$ in D. On the other hand, from the regularity of the boundary ∂D there follows that v(z) is continuous and equal to 0 at the points $z \in \partial D \setminus K$.

Assume now that U is the unit circle and $\varphi : U \to D$ is a uniformization of the domain D. From the regularity of ∂D and from a theorem of Nevanlinna [5, p. 214 of the Russian edition] there follows that φ is a function of bounded form. Consequently, φ has radial limits a.e. on ∂U. These radial limits are a.b.p. of the domain D. We consider the harmonic function $v(\varphi(z)), z \in U$. It is bounded in U and has radial limits, equal to 0 a.e. on ∂U. Consequently, $v \equiv 0$. The obtained contradiction proves the lemma.

LEMMA 4 [6]. Let $v \geqslant 0$ be a δ-subharmonic function in an arbitrary domain G. We set $E = \{z \in G : v(z) = 0\}$. Then the restriction of the Riesz charge of the function v to E is a nonnegative mesure.

From Iversen's subharmonic theorem there follows that the point ∞ is attainable from all the domains D_j [7, Sec. 4.6.5]. We denote by $\partial_0 D_j$ the set of the a.b.p. of the domain D_j.

LEMMA 5. $\mu_j(\partial D_j \setminus \partial_0 D_j) = 0$.

Proof. Let $R > 0$ be an arbitrarily large number. We consider the domain $G = D_j \cup \{z \in \overline{C} : |z| > 2R\}$. The domain G is regular since each point $z \in \partial G$ is contained in some continuum $K \subset \partial G$ [8, Chap. IX, Sec. 3]. Let g be the Green function of the domain G with pole at ∞, extended in $C \setminus G$ by taking it equal to zero. We select a constant $C > 0$ so that $u_j(z) < g(z)$, $|z| = 3R$, and we apply Lemma 4 to the functions $v = cg - u_j$ in the domain D(0, 2R). We obtain that the measure $\mu_j|_{D(0,R)}$ is absolutely continuous with respect to the harmonic measure ν of the domain G. Since a point $z \in G$, $|z| < R$ is unattainable from G if and only if it is unattainable from D_j, the assertion of Lemma 5 follows from Lemma 3.

LEMMA 6. Let E be a set of points, attainable simultaneously from three or more domains D_j. Then E is finite and $\mu_j(E) = 0$ for all j, $1 \leqslant j \leqslant n$.

Proof. Assume that D_1, D_2, D_3 have two common finite a.b.p. z' and z". Let $\Gamma_i \subset D_i$ be simple curves, joining z' and z", $1 \leqslant i \leqslant 3$. Then one of the three curves Γ_i (Γ_1, say) lies in the interior domain of the closed Jordan curve formed by the two other curves (Γ_2 and Γ_3, say) and the points z' and z". From here, by Jordan's theorem, taking into account that $D_1 \cap (\Gamma_2 \cup \Gamma_3 \cup \{z', z''\}) = \varnothing$, we obtain that the domain D_1 lies in the interior domain of the closed Jordan curve $\Gamma_2 \cup \Gamma_3 \cup \{z', z''\}$; this is not possible since D_1 is unbounded. Thus, card $\times$ $(\partial_0 D_1 \cap \partial_0 D_2 \cap \partial_0 D_3) \leqslant 2$ (here we have taken into account that the point ∞ is attained from all the domains D_j). The finitenes of the set E is proved. The second assertion of the lemma follows from the fact that $u_j \geqslant 0$. In this case $\mu_j(E) = 0$ for any finite set $E \subset C$.

We consider now the restrictions of measures

$$\mu_{ji} = \mu_j|_{D_j \cup (\partial_0 D_j \setminus \bigcup_{i \neq j} \partial_0 D_i)},$$

*In fact, the regularity condition is redundant (see [4]).

$$\mu_{ij} = \mu_j|_{\partial_0 D_i \cap \partial_0 D_j \setminus \{\infty\}}, \quad 1 \leqslant i, \ j \leqslant n.$$

From Lemmas 5 and 6 there follows that

$$\mu_k = \sum_{j=1}^{n} \mu_{kj}, \quad 1 \leqslant k \leqslant n. \tag{9.1}$$

We can assume that the measures μ_{ik} and μ_{mp} have nonintersecting Borel supports if the unordered pairs $\{i, k\}$ and $\{m, p\}$ do not coincide. Therefore, from (8.1) and (9.1) there follow the relations (8.3)-(8.5). In particular, the functions u_j are harmonic in D_j (8.4). It is easy to see that u_j are continuous in $\boldsymbol{C}$.

LEMMA 7. The harmonic measures on ∂D_j are absolutely continuous with respect to the measure μ_j.

Proof. Let $z_0 \in D_j$. We select $r > 0$ so that $D(z_0, r) \subset D_j$, and we set $K = \partial D(z_0, r)$. Let g be the Green function of the domain D_j with pole at z_0, extended in $\bar{\boldsymbol{C}} \setminus D_j$ by taking it equal to zero. We select a constant $C > 0$ so that $g(z) \leqslant C u_j(z), \ z \in K$. The application of Lemma 4 with $v = Cu_j - g$ and $G = \boldsymbol{C} \setminus D(z_0, r)$ concludes the proof.

10. Mutual Dispersion of the Domains D_j. We consider n copies U_j, $1 \leqslant j \leqslant n$, of the unit circle and we fix the conformal homeomorphisms $\varphi_j: U \to D_j$ such that the segments $(-1, 0] \subset U_j$ should go into curves exiting at ∞ (see the remark preceding Lemma 5). The radial limits of the function φ_j are a.b.p. of the domain D_j.

We show that the finite radial limits of the function φ_j at the points $x, y \in T_j = \partial U_j$, $x \neq y$, cannot coincide. Otherwise, we consider the curve γ, consisting of the two radii through the points x, y. The curve $\Gamma = \overline{\varphi_j(\gamma)}$ is a Jordan curve. The domain D, bounded by the curve Γ, does not intersect $\bar{D}_k$ for $k \neq j$ since all D_k are unbounded. On the other hand, inside D we have a part of ∂D_j of positive harmonic measure. From Lemma 7 there follows that $\mu_{jj}(D) > 0$, which contradicts (8.4).

We say that the domains D_i and D_j are contiguous if there exist at least two common boundary points, attainable from both domains. Obviously, if $\mu_{ij} \neq 0$, then D_i and D_j are contiguous. We fix a number j, $1 \leqslant j \leqslant n$. Assume that the domain D_i is contiguous to D_j. We consider the set $X_{ji} \subset T_j$, corresponding to the finite a.b.p. of the domain D_j, which are simultaneously a.b.p. of the domain D_i. We set

$$b_{ji} = \inf\{\theta \in (-\pi, \pi) : e^{i\theta} \in X_{ji}\};$$
$$a_{ji} = \sup\{\theta \in (-\pi, \pi) : e^{i\theta} \in X_{ji}\};$$
$$T_{ji} = (b_{ji}, a_{ji}) \subset T_j.$$

The arc T_{ji} is called the contiguity arc.

We show that none of the contiguity arcs T_{ji} contains points in which the radial limit of the function φ_j is infinite. Otherwise, we would have points $x, y, t, -\pi < x < y < t < \pi$, with radial limits $\varphi_j(e^{ix}) = a, \ \varphi_j(e^{iy}) = \infty, \ \varphi_j(e^{it}) = b$, and a and b are finite a.b.p. of the domain D_i. We join a and b simple arcs $\gamma_1 \subset D_j$, $\gamma_2 \subset D_i$. The closed Jordan curve $\Gamma = \gamma_1 \cup \gamma_2 \cup \{a, b\}$ divides the plane. The curve $\varphi_j^{-1}(\gamma_1)$ divides U_j into two parts, one of which has on the boundary the point -1, while the other the point e^{iy}. The images of both of these domains are unbounded, which is not possible since one of these images lies in a domain bounded by the curve Γ.

We show that the contiguity arcs T_{ji}, T_{jk} do not intersect for $i \neq k$. If these two arcs would intersect, then we would find three points $-\pi < x < y < t < \pi$, such that there would exist finite, mutually distinct radial limits $\varphi_j(e^{ix}) = a, \ \varphi_j(e^{iy}) = b, \ \varphi_j(e^{it}) = c$ and, moreover, the points a and c would be attainable from one domain (D_i, say), while the point b from another one (D_k, say). We join the points a and c by simple arcs in D_j and in D_i. We obtain a closed Jordan curve, bounding a domain D, and $b \in D$, since $b \neq a$, $b \neq c$. But the domain D_k, being unbounded, does not intersect D. We obtain a contradiction.

We show that the contiguity arcs T_{ji}, together with their endpoints, fill out the entire circumference T_j. Indeed, if the arc $\Delta \subset T_j$ does not intersect any of the contiguity arcs, then the finite radial limits of the function φ_j on Δ are not attainable boundary points for any of the domains D_k, $k \neq j$. Taking into account Lemma 7, we obtain a contradiction with (8.4).

A similar reasoning shows that the points on T_{ji}, in which there exist radial limits of the function φ_j, being a.b.p. of the domain D_i, are dense in the arc T_{ji}.

We supply the circumference T_j with a positive orientation. One has a natural monotone (changing the orientation to the opposite) mapping ψ_{ji} of a dense subset of the arc T_{ji} to a dense subset of the arc T_{ij}. This mapping takes the point $e^{ix} \in T_{ji}$ into the point $e^{iy} \in T_{ij}$, if the corresponding radial limits are equal: $\varphi_j(e^{ix}) = \varphi_i(e^{iy})$. We extend the indicated mapping to an arc homeomorphism. This can be done since the functions ψ_{ji} and $\psi_{ij} = \psi_{ji}^{-1}$ are strictly monotone. We paste together the circles U_j and U_i along the arcs T_{ji} and T_{ij}, identifying these arcs with the aid of the mapping ψ_{ji}. We carry this out for all pairs i, j for which the arcs T_{ij} and T_{ji} are defined. After pasting a finite number of punctures we obtain a compact oriented surface S. The topological imbeddings $\varphi_j^{-1}: D_j \to U_j \subset S$ are defined. On the surface S we have a net, consisting of the edges $T_{ij} = T_{ji}$ and the vertices are the pasted punctures.

A closed curve $\Gamma \subset S$ is said to be admissible if it does not pass through the vertices, intersects transversally the edges a finite number of times, at each such point of intersection $x \in \Gamma \cap T_{ij}$ there exist radial limits $\varphi_i(x)$, $\varphi_j(x)$, and Γ form nonzero angles with T_{ij}. An admissible curve $\Gamma \subset S$ has an image in $\bar{C}$ a closed curve $\bigcup_j \overline{\varphi_j(\Gamma \cap U_j)}$, which will be denoted by $\varphi(\Gamma)$. Obviously, the admissible curves are dense in the set of all closed curves on S.

We show that the surface S is homeomorphic to a sphere. If this is not so, then there exist two admissible curves Γ_1, Γ_2, intersecting transversally at a unique point, not lying on an edge. Then the closed curves $\varphi(\Gamma_1)$ and $\varphi(\Gamma_2)$ on the sphere intersect transversally at a unique point, which is not possible. Thus, S is a sphere.

Let Q be the set of all odd vertices of the net. The number of such vertices is even (see Sec. 8). We consider an arbitrary vertex p. Let Γ_m be a sequence of admissible closed Jordan curves, converging to p and such that $\Gamma_{m+1} \cap \Gamma_m = \varnothing$, Γ_{m+1} separates Γ_m from p. We denote by K_m that component of the set $\bar{C} \setminus \varphi(\Gamma_m)$ which contains Γ_{m+1}. We have $K_{m+1} \subset K_m$ and, therefore, $\bigcap_{m=1}^{\infty} K_m$ is a nonempty set, which we denote by K(p). The curves $\varphi(\Gamma)$, where $\Gamma \subset S$ is an admissible curve, do not intersect with the sets K(p). For distinct vertices $p, q \in S$ we have $K(p) \cap K(q) = \varnothing$. For each odd vertex q we select an arbitrary point $\varphi(q) \subset K(q)$. For any admissible curve Γ we have

$$\operatorname{ind}_q \Gamma = \operatorname{ind}_{\varphi(q)} \varphi(\Gamma). \tag{10.1}$$

We consider a two-sheeted covering $\pi: F \to \bar{C}$ of the sphere $\bar{C}$ by some Riemann surface F, ramified exactly over the points $\varphi(q)$, $q \in Q$. Let $\tilde{D}_1, \ldots, \tilde{D}_{2n}$ be the preimages of the domains $D_1, \ldots, D_n$, and, moreover, $\pi^{-1}(D_j) = \tilde{D}_j \cup \tilde{D}_{n+j}$. The definition of the contiguity of domains on the Riemann surface F is exactly the same as in $\bar{C}$. If the domains $\tilde{D}_i$ and $\tilde{D}_j$ are contiguous, then also $\pi(D_i)$ and $\pi(D_j)$ are contiguous, but the converse is not true.

<u>LEMMA 8.</u> Let $\tilde{D}_{j_1}, \ldots, \tilde{D}_{j_m}$ be a finite sequence and, moreover, D_{j_k} is contiguous to $D_{j_{k+1}}$, $1 \leqslant k \leqslant m$, $j_{m+1} = j_m$. Then m is even.

<u>Proof.</u> Let $\pi(\tilde{D}_{j_k}) = D_{i_k}$. The domains D_{i_k} and $D_{i_{k+1}}$ are contiguous. Therefore, in U_{i_k} there exists an arc Γ_k, joining the points $x \in T_{i_k i_{k-1}}$ and $y \in T_{i_k i_{k+1}}$ and, moreover, the functions φ_{i_k} and $\varphi_{i_{k-1}}$ have radial limits at the point x, while φ_{i_k} and $\varphi_{i_{k+1}}$ at the point y. We select the arc Γ_k so that the function φ_{i_k} should have limit values along Γ_k at the points x and y. The successively passed arcs Γ_k form an admissible curve Γ and, moreover, $m = n(\Gamma)$. Obviously, the curve $\varphi(\Gamma)$, is lifted to the Riemann surface F. Therefore,

$$\sum_{q \in Q} \operatorname{ind}_{\varphi(q)} \varphi(\Gamma) \equiv 0 \pmod 2. \tag{10.2}$$

From (10.1), (10.2) there follows that

$$\sum_{q \in Q} \operatorname{ind}_q \Gamma \equiv 0 \pmod 2.$$

Therefore the number $m = n(\Gamma)$ is even by virtue of Lemma 2.

<u>11. Conclusion of the Proof.</u> It remains to repeat the arguments of Sec. 8. We lift the functions u_j to F and we obtain subharmonic functions $\tilde{u}_j$ with supports in $\tilde{D}_j$. We partition the domains $\tilde{D}_j$ into two classes (0th and 1st) so that contiguous domains should belong to distinct classes. This can be done by virtue of Lemma 8. We denote by k(j) the class of the domain $\tilde{D}_j$ and we set

$$v = \sum_{j=1}^{2n} (-1)^{k(j)} \tilde{u}_j.$$

The Riesz charge of this function is equal to $\sum_{j,i} (-1)^{k(j)} \tilde{\mu}_{ji}$. This sum is equal to 0 since contiguous domains belong to distinct classes and $\tilde{\mu}_{ji} = \tilde{\mu}_{ij}$ by virtue of (8.4). Thus, v is a harmonic function, while the domains D_j are bounded by piecewise analytic curves. The reasoning from Sec. 8 with the use of (8.8), (8.9) shows that

$$v(z) = \operatorname{Re} az^{n/2}, \ a \in C, \ z \subset C,$$

$n \geqslant 2$ is a natural number. The fundamental lemma is proved.

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PAIRS OF REGULARIZABLE INVERTIBLE OPERATORS WITH A NONREGULARIZABLE SUPERPOSITION

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Let X and Y be Banach spaces. The set of continuous linear injective operators, acting from X into Y, will be denoted by $L_0(X, Y)$, while in the case X = Y, simply by $L_0(X)$. Clearly, for $A \in L_0(X, Y)$ there exists an inverse operator A^{-1} which, in general, is not defined on all of Y but only on the range of the operator A, and need not be continuous. In the theory of ill-posed problems, from these operators one isolates the subclass of regularizable operators.

Definition 1 [1, p. 179]. Let $A \in L_0(X, Y)$. A sequence of mappings $R_n: Y \to X$ $(n \in N)$ is said to be a regularizer for the operator A^{-1} if for each $x \in X$ we have the relation

$$\lim_{n \to \infty} \sup \{ \|x - R_n y\| : y \in Y, \ \|y - Ax\| \leqslant 1/n \} = 0.$$

If the operator A^{-1} has a regularizer, then it is said to be regularizable.

Definition 2. We shall say that a Banach space X has the property of the regularizability of superpositions (denoted $X \in RS$) if for any $A, B \in L_0(X)$, from the regularizability of A^{-1} and B^{-1} there follows the regularizability of $(BA)^{-1}$.

In [2] it is proved that $c_0 \notin RS$. On the other hand, it is known [1, p. 193] that in a quasireflexive Banach space (i.e., a space, whose canonical image in the second conjugate space has a finite codimension), for any $A \in L_0(X)$ the inverse opertor A^{-1} is regular and, consequently, all the quasireflexive spaces possess the RS property.

This paper is devoted to the further investigation of the RS property.

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