EXCEPTIONAL VALUES IN THE R. NEVANLINNA SENSE

AND IN THE V. P. PETRENKO SENSE. 1

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1. Without explanation we shall use the standard notations of the Nevanlinna theory of the distribution of the values of meromorphic functions (see, for example, [1]). In the classical theory of value distribution, as a measure for the nearness of a function f, meromorphic in $\mathbf{C}$, to the value $a \in \mathbf{C}$, one selects the quantity m(r, a, f), i.e., except for the factor $1/2\pi$, the norm $\|\ln^+|f(re^{i\varphi}) - a|^{-1}\|$ in the space L_1 of the function $\ln^+|f - a|^{-1}$, as a function of $\varphi \in [0, 2\pi]$. In the last 15 years norms in other spaces have also been used; the norm in the space L_∞ has a special importance. If $M(r, f) = \max\{|f(z)|:|z| = r\}$, then as the measure of the nearness of f to a one can take $\ln^+ M(r, 1/(f - a)) = L(r, a, f)$. In 1969, with the aid of $L(r, a, f)$, V. P. Petrenko has introduced the quantity

$$\beta(a, f) = \lim_{r\to\infty} L(r, a, f)/T(r, f),$$

called by him the deviation of the meromorphic function f relative to the number a. In a series of papers, V. P. Petrenko has investigated in detail the properties of $\beta(a, f)$, the fundamental results being summarized in [2]. The set $\{a \in \bar{\mathbf{C}} : \delta(a, f) > 0\}$ of R. Nevanlinna deficiency values will be denoted by $E_N(f)$, while the set $\{a \in \bar{\mathbf{C}} : \beta(a, f) > 0\}$ of positive deviations by $E_\Pi(f)$. Directly from the definitions there follows $\delta(a, f) \le \beta(a, f)$ and, therefore, $E_N(f) \subset E_\Pi(f)$. Petrenko has proved that, for meromorphic functions of finite lower order, $E_\Pi(f)$ is at most countable and has proposed to elucidate whether for functions of finite lower order one has always $E_N(f) = E_\Pi(f)$; that this is not so for functions of finite lower order has been elucidated by Petrenko himself [2, Theorem 3.1.1]. In 1976, Grishin [3; 2, Theorem 3.1.1] has shown that for each ρ, $0 \le \rho < \infty$ there exists a meromorphic function f of order ρ, for which $\delta(\infty, f) = 0$ while $\beta(\infty, f) > 0$. Grishin's construction has been appreciably simplified [4; 2, Theorem 2.4.2]. Then one has constructed an example of a meromorphic function f of any given order ρ, $0 < \rho < \infty$, for which $E_N(f) = \phi$, while $E_\Pi(f)$ is an arbitrary, at most countable subset of $\bar{\mathbf{C}}$ [5]. Thus, one has obtained an answer to one of the questions of Petrenko [2, pp. 8 and 73]. It is natural to formulate the question of the complete description of the relationship between the sets $E_N(f)$ and $E_\Pi(f)$ for meromorphic functions of finite order ρ [6, No. 11.3]. If $\rho = 0$, then it is known that $E_N(f)$ and $E_\Pi(f)$ consist of at most one point (according to Valiron's theorem [1, pp. 90 and 158] and Petrenko's theorem [2, Theorem 2.6.1], respectively). Together with Grishin's example (quoted above), this gives the complete description of the sets $E_N(f)$ and $E_\Pi(f)$ for order zero. For a positive order the answer is given by the following theorem.

THEOREM 1. Let $0 < \rho < \infty$, $E_1 \subset E_2 \subset \bar{\mathbf{C}}$, E_2 being at most countable. There exists a meromorphic function f of order ρ, for which $E_N(f) = E_1$, $E_\Pi(f) = E_2$.

In [7], it is proved that for meromorphic functions having the property

$$T(2r, f) = O(T(r, f)),\ r \to \infty, \tag{1.1}$$

we always have $E_N(f) = E_\Pi(f)$. Thus, the inequality $E_N \ne E_\Pi(f)$ is connected with a definite nonregularity of the growth. Nevertheless, the functions f from Theorem 1 can be selected so that for them the lower order λ be equal to the order ρ.

In the examples of [3-5], the function f with $E_N(f) \ne E_\Pi(f)$ is meromorphic. There arises the question whether $E_N(f) \ne E_\Pi(f)$ is possible for entire functions of finite order [6, No. 11.3]. For $\rho \le 1/2$ for an entire function f it is known that $E_N(f) = \{\infty\}$ [1, p. 269] and $E_\Pi(f) = \{\infty\}$ [1, p. 273]. For $\rho > 1/2$ the answer is the following.

THEOREM 2. For each ρ, $1/2 < \rho < \infty$, there exists an entire function f of order ρ for which $\delta(0, f) = 0$, $\beta(0, f) > 0$.

Translated from Teoriya Funktsii, Funktsional'nyi Analiz i Ikh Prilozheniya, No. 47, pp. 41-51, 1987. Original article submitted September 16, 1985.

Theorems 1 and 2 have been communicated by the authors in the note "Deficiencies and deviations of meromorphic functions of a finite order," Dokl. Akad. Nauk Ukr.SSR, No. 10, 3-5 (1984). In a letter dated April 30, 1984, D. Drasin communicated to the authors that he and his student R. Gillespie have solved Problem 11.3 of [6], independently from the authors. Their examples are based on different ideas.

The methods used here for the proof of Theorems 1 and 2 allow us to obtain also other results of interest.

THEOREM 3. For each ρ, $1/2 < \rho < \infty$, there exists an entire function f of order ρ such that $\delta(0, f) > 0$ and (1.1) is not satisfied.

Kotman [9] has proved this theorem for $\rho > 1$, giving at the same time a negative answer to a question raised by Hayman [10, No. 1.10]. Theorem 3 closes this question completely since, as noted, for $\rho \le 1/2$ one has necessarily $\delta(0, f) = 0$.

According to the mentioned communication of Drasin, he and Gillespie have proved Theorem 3 with $\delta(0, f) \ge 1/2$ by another method.

We shall consider the question of the dependence of the deficiencies of an entire function of finite order on the selection of the origin, i.e., whether $\delta(0, f(z)) = \delta(0, f(z + h))$, $h \in \mathbf{C}$ is always true. It is known [1, Chap. IV, Sec. 6] that the deficiency may vary when passing from f(z) to f(z + h) for meromorphic functions of infinite order (Dugué, 1947), for entire functions of infinite order (Hayman, 1953), for meromorphic functions of finite order (Gol'dberg, 1954). Recently, Miles [11] constructed an example of an entire function f of "very large finite order ρ," for which $0 = \delta(0, f(z)) < \delta(0, f(z + h))$, $h \ne 0$. In Miles's example, the upper estimation of the order ρ is difficult because of the necessity of solving a cumbersome system of inequalities involving transcendental functions. Here we shall construct an example of a function, simpler than that of Miles, for which the order is relatively small.

THEOREM 4. Let $5 < \rho < \infty$. There exists an entire function f of order ρ such that $0 = \delta(0, f(z - 1)) < \delta(0, f(z))$.

If $\rho < 3/2$, then for an entire function the deficiency does not depend on the selection of the origin [1, p. 232, Corollary 2]. The question is open for $\rho \in [3/2, 5]$.

We introduce some notations. By S we shall denote the class of subharmonic functions in $\mathbf{C}$. If $u \in S$, then $B(r, u) = \max\{u(z): |z| = r\}$, $A(r, u) = \inf\{u(z): |z| = r\}$.

By $e_m(x)$ and $\ln_m x$ we denote the m-th iterates of the functions $\exp x$ and $\ln x$. Further,

$$\|f(\theta)\| = \frac{1}{2\pi}\int_0^{2\pi} |f(\theta)|\, d\theta.$$

The symbol $\partial/\partial n$ denotes the derivative in the direction of the interior normal to the boundary of a domain. By x^+ and x^- we denote $(|x| + x)/2$ and $(|x| - x)/2$, respectively.

2. Proof of Theorem 1. Without loss of generality, we can assume that $E_2 \subset \mathbf{C}$. Taking into account the result of [5], here we restrict ourselves to the case when $E_1 \ne \phi$. For meromorphic functions f of finite lower order we have $E_\Pi(f) \subset E_V(f)$, where $E_V(f)$ is the set of exceptional values for f in the sense of Valiron [2, p. 52]. For any at most countable set E_1 one knows examples of meromorphic functions f of order ρ, $0 < \rho < \infty$, for which $E_N(f) = E_V(f) = E_1$ [1, pp. 161-166]. Then also $E_\Pi(f) = E_1$. Therefore, the case $E_1 = E_2$ will not be considered. Thus, $E_1 \ne \phi$, E_2. We shall assume that $0 < \rho < 1/2$. The general case is obtained at once if we note that for $f_1(z) = f(z^n)$ we have $\delta(a, f_1) = \delta(a, f)$, $\beta(a, f_1) = \beta(a, f)$, while the order of f_1 is equal to $n\rho$, where ρ is the order of f, $n \in \mathbf{N}$.

LEMMA 2.1. There exists $v \in S$ such that: 1) $v(r) = B(r, v) := \lambda(r)$, $r \ge 0$, $\lambda(0) = 0$; 2) $\ln\lambda(r) \sim \rho\ln r$, $r \to \infty$; 3) there exist sequences (r_n'), (r_n''), (c_n) such that $r_n' \to \infty$, $r_n''/r_n' \to \infty$, $\lambda(r) \sim c_n r^\rho$, $r_n'' \le r \le r_n''$, $n \to \infty$; 4) $v(re^{i\theta}) \le \lambda(r)\cos\rho\theta$, $r \ge 0$, $|\theta| \le \pi$; 5) $v(re^{i\theta}) = (1 + O(1))\lambda(r)\cos\rho\theta$, $|\theta| \le \pi$, $r \to \infty$, $r \in \bigcup_{n=1}^{\infty}[r_n', r_n'']$; 6) there exist a sequence R_n, $r_{n-1}'' < R_n < r_n'$ and a sequence $\gamma_n \downarrow 0$ such that for all $re^{i\theta}$, $1 + (\ln_2 R_n)^{-1} \le r/R_n \le 1 + 2(\ln_2 R_n)^{-1}$, $\gamma_n \le |\theta| \le \pi$, we have $v(re^{i\theta}) = o(\lambda(r))$, $r \to \infty$.

Proof. We set

$$G(re^{i\theta}) = (r^\rho - 2^\rho)^+ \cos\rho\theta, \quad 0 \leqslant r < \infty, \quad |\theta| \leqslant \pi,$$
$$H(re^{i\theta}) = \begin{cases} r^2\cos 2\theta, & |\theta| \leqslant \pi/4, \ 0 \leqslant r < \infty, \\ 0, & \pi/4 \leqslant |\theta| \leqslant \pi, \ 0 \leqslant r < \infty. \end{cases}$$

It is easy to see that $G \in S$, $H \in S$. Let

$$u(re^{i\theta}) = \begin{cases} CG(re^{i\theta}), & r \geqslant 3, \\ \max\{CG(re^{i\theta}),\ H(re^{i\theta}-1)\}, & r < 3, \end{cases}$$

where the constant $C > 2$ is so large that for $r \in [2.5;\ 3]$ we have $H(re^{i\theta} - 1) < CG(re^{i\theta})$. Then $u \in S$. We set

$$R_n = e_3(n), \quad A_n = R_n^\rho \ln R_n, \quad v(z) = \sum_{k=1}^{\infty} A_k u(z/R_k).$$

Since $u(z) = 0$ for $|z| \leq 1$, it follows that this series reduces in each circle $\{z\colon |z| \leq R\}$ to a finite sum and $v \in S$. We show that for v the assertions of Lemma 2.1 hold. Statement 1 is obvious. Let $r \in [R_n, R_{n+1}]$. Then

$$\lambda(r) = \sum_{k=1}^{n-1} A_k C\left\{\left(\frac{r}{R_k}\right)^\rho - 2^\rho\right\} + A_n u\left(\frac{r}{R_n}\right).$$

Since for sufficiently large n we have $R_n/R_{n-1} > 2^{1+1/\rho}$, it follows that

$$\lambda(r) \geqslant A_{n-1}C\{(r/R_{n-1})^\rho - 2^\rho\} > 0.5A_{n-1}C(r/R_{n-1})^\rho = 0.5Cr^\rho \ln R_{n-1} > r^\rho,$$

and, therefore,

$$\varliminf_{r\to\infty} \frac{\ln\lambda(r)}{\ln r} \geqslant \rho. \tag{2.1}$$

If $r \in [R_n \cdot 2.5R_n]$, then

$$\lambda(r) \leqslant \sum_{k=1}^{n-1} A_k C(r/R_k)^\rho + A_n u(2.5) \leqslant Cr^\rho \sum_{k=1}^{n-1} \ln R_k + u(2.5)R_n^\rho \ln R_n \leqslant (2C + u(2.5))\, r^\rho \ln r.$$

If $r \in [2.5R_n,\ R_{n+1}]$, then

$$\lambda(r) \leqslant \sum_{k=1}^{n} A_k C(r/R_k)^\rho \leqslant 2Cr^\rho \ln R_n \leqslant 2Cr^\rho \ln r.$$

Consequently,

$$\varlimsup_{r\to\infty} \frac{\ln\lambda(r)}{\ln r} \leqslant \rho. \tag{2.2}$$

From (2.1) and (2.2) it follows that 2 holds.

For $r \in [R_n^2,\ R_{n+1}]$,

$$\sum_{k=1}^{n-1} A_k C((r/R_k)^\rho - 2^\rho) \leqslant \sum_{k=1}^{n-1} Cr^\rho \ln R_{n-1} = o(r^\rho \ln R_n), \quad r\to\infty, \tag{2.3}$$
$$\lambda(r) = (1+o(1))\,A_n C((r/R_n)^\rho - 2^\rho) = (1+o(1))\,Cr^\rho \ln R_n, \quad r\to\infty. \tag{2.4}$$

We set $r_n' = R_n^2$, $r_n'' = R_{n+1}$, $c_n = C\ln R_n$. Then from (2.4) follows 3. From (2.3) and (2.4) it follows that for $r \in [r_n', r_n'']$, $|\theta| \leq \pi$, $n \to \infty$ we have

$$v(re^{i\theta}) = o(r^\rho \ln R_n) + A_n CG(re^{i\theta}/R_n) =$$
$$= (1+o(1))\,A_n C((r/R_n)^\rho - 2^\rho)\cos\rho\theta = (1+o(1))\,\lambda(r)\cos\rho\theta,$$

i.e., 5 holds.

For $z = re^{i\theta}$, $r \geq 1$, $|\theta| \leq \pi$, $|\theta'| = |\arg(z-1)| \leq \pi/4$ we have

$$H(z-1) = |z-1|^2\cos 2\theta' \leqslant (r-1)^2\cos 2\theta'/\cos^2\theta' = (r-1)^2(1-\operatorname{tg}^2\theta') \leqslant (r-1)^2\cos\theta' \leqslant (r-1)^2\cos\rho\theta = B(r, H(z-1))\cos\rho\theta,$$

for $\pi/4 \leq |\theta'| \leq \pi$ this inequality is obvious. Now it is easy to show that $u(re^{i\theta}) \leq B(r, u)\cos\rho\theta$, $|\theta| \leq \pi$, and then statement 4 follows.

We prove the validity of 6. Let $r = R_n(1 + t/\ln_2 R_n)$, $t \in [1, 2]$. The circumference $\{z: |z| = r/R_n\}$ intersects the ray $\{z: \arg(z-1) = \pi/4\}$ at the point $(r/R_n)e^{i\gamma(t)}$. It is easy to compute that $\gamma(t) = \pi/4 - \arcsin\{\sqrt{2}(1 + t/\ln_2 R_n)\}^{-1}$. We set $\gamma_n = \gamma(2) = O((\ln_2 R_n)^{-1})$, $n \to \infty$. Since $\gamma(t) \leq \gamma(2)$, $t \in [1; 2]$, for $\gamma_n \leq |\theta| \leq \pi$, $k \geq n$ we have $u(re^{i\theta}/R_k) = 0$. Therefore,

$$v(re^{i\theta}) \leqslant \sum_{k=1}^{n-1} A_k C(r/R_k)^\rho = (1 + o(1))Cr^\rho \ln R_{n-1} = (1 + o(1))CR_n^\rho \ln R_{n-1},\ n \to \infty. \tag{2.5}$$

On the other hand, for the considered r we have

$$\lambda(r) = v(r) \geqslant A_n u(r/R_n) = R_n^\rho \ln R_n t^2(\ln_2 R_n)^{-2} \geqslant 10^{-4} R_n^\rho \ln R_n/(\ln_2 R_n)^2. \tag{2.6}$$

Since $\ln R_{n-1} = o(\ln R_n/(\ln_2 R_n)^2)$, $n \to \infty$, from (2.5) and (2.6) follows statement 6.

The following lemma is proved in [1, pp. 207-208].

LEMMA 2.2. Let $\lambda(r)$ be a nondecreasing, logarithmically convex function, whose order $\rho < 1$. Then

$$\lim_{r\to\infty} \lambda(r + O(\ln r))/\lambda(r) = 1. \tag{2.7}$$

LEMMA 2.3. Let λ be the function from Lemma 2.1 and let $v_1(re^{i\theta}) = \lambda(r)\cos\rho\theta$, $0 \leq r < \infty$, $|\theta| \leq \pi$. Then $v_1 \in S$.

Proof. Since

$$v_1(re^{i\theta}) = \sum_{k=1}^{\infty} A_k u(r/R_k)\cos\rho\theta,$$

it is sufficient to show that $u(r)\cos\rho\theta \in S$. We have $u(r) = 0$ for $0 \leq r \leq 1$,

$$u(r) = \max\{(r-1)^2,\ C(r^\rho - 2^\rho)^+\},\ 1 \leqslant r \leqslant 3,$$
$$u(r) = C(r^\rho - 2^\rho),\ r \geqslant 3.$$

But $C(r^\rho - 2^\rho)^+\cos\rho\theta \in S$ and $(r-1)^2\cos\rho\theta \in S$ for $r > (1+\sqrt{2})/4$, since

$$\Delta\{(r-1)^2\cos\rho\theta\} = r^{-2}(4r^2 - 2r - \rho^2)\cos\rho\theta > r^{-2}(4r^2 - 2r - 4^{-1})\cos\rho\theta > 0, \tag{2.8}$$

while the subharmonicity in the neighborhood of the negative ray can be easily verified. Thus, $u(r)\cos\rho\theta \in S$.

Definition. We say that a set $E \subset \mathbf{C}$ belongs to the class (σ) if it is covered by circles with a finite sum of the radii.

We denote by $S_\rho(\theta)$, $0 < \rho < 1/2$, the 2π-periodic function $S_\rho(\theta) = \cos\rho\theta$ for $|\theta| \leq \pi$.

LEMMA 2.4. Suppose that the conditions of Lemma 2.1 hold. There exist entire functions F and G and $E \in (\sigma)$ such that

$$\ln M(r, G) \sim \ln M(r, F) \sim \lambda(r),\ r \to \infty, \tag{2.9}$$

$$\ln|G(re^{i\theta})| \sim S_\rho(\theta)\lambda(r),\ re^{i\theta} \notin E,\ r \to \infty, \tag{2.10}$$

$$\ln|G(re^{i\theta})| \leqslant (1+o(1))\, S_\rho(\theta)\,\lambda(r),\ r\to\infty, \tag{2.11}$$
$$\ln|F(re^{i\theta})| \leqslant (1+o(1))\, S_\rho(\theta)\,\lambda(r),\ r\to\infty. \tag{2.12}$$

There exist sequences (p_n), (q_n), $n \geq n_0$, $p_n \in [R_n(1 + (\ln_2 R_n)^{-1}, R_n(1 + 2(\ln_2 R_n)^{-1}]$, $q_n \in [r_n', r_n'']$, $r_n' = o(q_n)$, $q_n = o(r_n'')$, $n \to \infty$, such that the neighborhoods $\{z: |z| = p_n\}$ and $\{z: |z| = q_n\}$ do not intersect with E and

$$\ln|F(p_ne^{i\theta})| = o(\lambda(p_n)),\ \gamma_n \leqslant |\theta| \leqslant \pi,\ n\to\infty, \tag{2.13}$$
$$\ln|F(q_ne^{i\theta})| \sim S_\rho(\theta)\,\lambda(q_n),\ n\to\infty. \tag{2.14}$$

There exist functions φ_1 and φ_2, $\varphi_1(r) \to 0$ and $\varphi_2(r) \to 0$ for $r \to \infty$, such that

$$\ln|F(re^{i\varphi_1(r)})| \sim \lambda(r),\ r\to\infty, \tag{2.15}$$
$$\ln|G(re^{i\varphi_2(r)}| \sim \lambda(r),\ r\to\infty. \tag{2.16}$$

Proof. Here and in the sequel we shall use the following theorem of Yulmukhametov [12].

THEOREM Y. Let $u \in S$, $B(r, u) = O(r^\rho)$, $r \to \infty$, $\rho < \infty$. Then there exist an entire function f and $E \in (\sigma)$ such that

$$|u(z) - \ln|f(z)|| = O(\ln^2|z|)\, z\to\infty,\ z\notin E. \tag{2.17}$$

By Theorem Y one can find entire functions F and G for which outside some $E \in (\sigma)$ we have

$$|v(z) - \ln|F(z)|| = O(\ln^2|z|),\ z\to\infty,\ z\notin E, \tag{2.18}$$

$$|v_1(z) - \ln|G(z)|| = O(\ln^2|z|),\ z\to\infty,\ z\notin E. \tag{2.19}$$

From (2.19) there follows at once (2.10), while from (2.18) and property 5 of Lemma 2.1 we obtain (2.14). From (2.18) and (2.19) it follows that (2.9) holds first for $r \notin E_1$, where $E_1 = \{r > 0: \exists\, re^{i\theta} \in E\}$. Since $E_1 \in (\sigma)$, by virtue of the monotonicity of $\ln M(r, G)$, $\ln M(r, F)$, and $\lambda(r)$ and Lemma 2.2, relation (2.9) is satisfied for $r \to \infty$ without restrictions. From $E \in (\sigma)$, Lemma 2.2, (2.10), and the principle of maximum modulus there follows (2.11). From the same considerations and property 4 of Lemma 2.1 we obtain (2.12). Since $R_n(1 + 2/\ln_2 R_n) - R_n(1 + 1/\ln_2 R_n) > \{R_n(1 + 2/\ln_2 R_n)\}^{1/2}$, from property 6 of Lemma 2.1 and (2.18) there follows (2.13). From (2.9), (2.11), and (2.12) there follow at once (2.15) and (2.16).

Lemma 2.4 is proved.

We proceed now directly to the proof of Theorem 1. We select a sequence (a_k), $k \in \mathbb{Z}$, so that $a_{-k} = a_k$, $E_1 = \{a_{2m}: m \in \mathbb{Z}_+\}$, $E_2\backslash E_1 = \{a_{2m+1}: m \in \mathbb{Z}_+\}$. We select a sequence $\theta_k \uparrow \pi$, $k \to +\infty$, such that $\theta_{-k} = -\theta_k$, $\theta_0 = 0$, $\theta_{k+1} - \theta_k \downarrow 0$, $0 \leq k \to +\infty$, $\theta_{2k+1} = (\theta_{2k} + \theta_{2k+2})/2$. We set

$$H_k = \begin{cases} G, & k = 2m, \\ F, & k = 2m+1, \end{cases}$$

where the entire functions G and F are constructed in Lemma 2.4. We select a sequence (c_k), $c_k > 0$, $c_{-k} = c_k$, such that $\sum_{k=-\infty}^{\infty} c_k < \infty$, $\sum_{k=-\infty}^{\infty} c_k|a_k| < \infty$. Let

$$f_j(z) = \sum_{k\neq j} c_k(a_k - a_j)\, H_k(ze^{-i\theta_k}),$$
$$g_j(z) = \sum_{k\neq j} c_k H_k(ze^{-i\theta_k}).$$

The following lemma is derived in [5, pp. 202-203] from Cartan's identity [1, p. 33].

LEMMA 2.5. Assume that there are given an arbitrary sequence (R_q) of numbers and an arbitrary sequence (f_k) of meromorphic functions. Then the set

$$E = \{\theta \in [-\pi, \pi] : \forall k, q \geqslant 1$$
$$m(R_q, (f_k - e^{i\theta})^{-1}) < 2^{k+q+1}\pi \ln 2\}$$

is not empty.

Applying this lemma, we conclude that there exists a number $\alpha \in [0, 2\pi]$ such that for all $j \in \boldsymbol{Z}$, $n \in \boldsymbol{N}$ we have

$$m(p_n, e^{i\alpha}, f_j) \leqslant 2^{|j|+n+2}\pi \ln 2. \tag{2.20}$$

Assume further that

$$\psi_1(z) = \sum_{k=-\infty}^{\infty} c_k a_k H_k(ze^{-i\theta_k}) - e^{i\alpha},$$
$$\psi_2(z) = \sum_{k=-\infty}^{\infty} c_k H(ze^{-i\theta_k}), \quad f(z) = \psi_1(z)/\psi_2(z).$$

From here, by virtue of (2.9), for $r \to \infty$ we have

$$T(r, f) \leqslant T(r, \psi_1) + T(r, \psi_2) + O(1) \leqslant \ln M(r, \psi_1) + \ln M(r, \psi_2) + O(1) \leqslant (2 + o(1))\lambda(r). \tag{2.21}$$

Let $k(\theta) = \sup\{S_\rho(\theta + \theta_m): m \in \boldsymbol{Z}\}$. It is easy to see that for all $\theta \in [0, \pi) \cup (\pi, 2\pi)$, in the definition of $k(\theta)$ one can take max instead of sup, $0 < k(\theta) < 1$, $k(\theta_m) = 1$, $m \in \boldsymbol{Z}$, $k(\pi) = 1$. The continuous function k has angular points at $\theta_{m+1}' = (\theta_m + \theta_{m+1})/2$. We set $\varphi_{2m+1}(r) = \varphi_1(r)$, $\varphi_{2m}(r) = \varphi_2(r)$, $m \in \boldsymbol{Z}$, where φ_1 and φ_2 are taken from (2.15) and (2.16).

We show that $\beta(a_j, f) > 0$. We have

$$f(z) - a_j = \frac{f_j(z) - e^{i\alpha}}{g_j(z) + c_j H_j(ze^{-i\theta_j})}. \tag{2.22}$$

From (2.11) and (2.12) if follows that for $r \to \infty$ we have

$$|f_j(re^{i(\theta_j+\varphi_j(r))})| \leqslant \exp\{(\omega_j + o(1))\lambda(r)\},$$
$$|g_j(re^{i(\theta_j+\varphi_j(r))})| \leqslant \exp\{(\omega_j + o(1))\lambda(r)\},$$

where $\omega_j = \max\{S_\rho(\theta_j - \theta_m): m \in \boldsymbol{Z}\setminus\{j\}\} = \max\{S_\rho(\theta_j - \theta_{j-1}), S_\rho(\theta_j - \theta_{j+1})\}$, $0 < \omega_j < 1$. On the other hand, by virtue of (2.15) and (2.16) we have

$$|c_j H_j(re^{i\varphi_j(r)})| \geqslant \exp\{(1 + o(1))\lambda(r)\}, \quad r \to \infty.$$

From (2.22) we obtain, therefore, that ($r \to \infty$)

$$|f(re^{i(\theta_j+\varphi_j(r))}) - a_j| \leqslant \exp\{(\omega_j - 1 + o(1))\lambda(r)\},$$
$$L(r, a_j, f) \geqslant (1 - \omega_j + o(1))\lambda(r).$$

Together with (2.21), this yields $\beta(a_j, f) \geq (1 - \omega_j)/2 > 0$.

Now we prove that $\delta(a_{2k}, f) = 0$. We set $j = 2k$ and we select $\eta = \eta_j$, $0 < \eta < \theta_{j+1}' - \theta_j$ for $j \geq 0$ and $0 < \eta < \theta_j - \theta_{j-1}'$ for $j < 0$. We consider the angle $W_j = \{z: \theta_j' + \eta \leq \arg z \leq \theta_{j+1}' - \eta\}$. Let $E^j = \{z: ze^{-i\theta_j} \in E\}$, where E is a set from (2.10). Then by (2.10) for $z = re^{i\theta} \in W_j\setminus E^j$, $r \to \infty$, we have

$$|H_j(ze^{-i\theta_j})| \geqslant \exp\{(S_\rho(\theta - \theta_j) + o(1))\lambda(r)\}. \tag{2.23}$$

Let

$$\omega_j(\theta) = \max\{S_\rho(\theta - \theta_m): m \in \boldsymbol{Z}\setminus\{j\}\} = \max\{S_\rho(\theta - \theta_{j-1}),$$
$$S_\rho(\theta_{j+1} - \theta)\}, \quad \theta_j' + \eta \leqslant \theta \leqslant \theta_{j+1}' - \eta.$$

It is easy to see that

$$\max\{\omega_j(\theta) - S_\rho(\theta - \theta_j): \theta_j' + \eta \leqslant \theta \leqslant \theta_{j+1}' - \eta\} =$$
$$= \max\{\omega_j(\theta) - S_\rho(\theta - \theta_j): \theta = \theta_j' + \eta, \ \theta = \theta_{j+1}' - \eta\} = -\gamma(j) < 0.$$

From (2.11) and (2.12) it follows that for $z = re^{i\theta} \in W_j$, $r \to \infty$, we have

$$|f_j(z)| \leqslant \exp\{(\omega_j(\theta) + o(1))\lambda(r)\}, \tag{2.24}$$

$$|g_j(z)| \leqslant \exp\{(\omega_j(\theta) + o(1))\lambda(r)\}. \tag{2.25}$$

From (2.22)-(2.25) we obtain that for $z = re^{i\theta} \in W_j \backslash E^j$ we have

$$|f(z) - a_j| \leqslant \exp\{(\omega_j(\theta) - S_\rho(\theta - \theta_j) + o(1))\lambda(r)\} \leqslant \exp\{(-\gamma(j) + o(1))\lambda(r)\},\ r \to \infty. \tag{2.26}$$

Together with (2.21) this gives

$$\delta(a_j, f) \geqslant \gamma(j)(\theta'_{j+1} - \theta'_j - 2\eta)/4\pi > 0,\ j = 2k.$$

From (2.26) it follows also that for $r \to \infty$ we have

$$T(r, f) \geqslant m(r, a_0, f) + O(1) \geqslant (L + o(1))\lambda(r),\ L > 0 \tag{2.27}$$

From (2.21), (2.27) and property 2 in Lemma 2.1 it follows that $\ln T(r, f) \sim \rho \ln r$, $r \to \infty$.

Now we show that $\delta(a_{2k}, f) = 0$. We set $j = 2k + 1$ and we fix η, $0 < \eta < \pi/2$. Let $k_0 > 0$ be such that $\theta_{2k_0-1} \leq \pi - \eta < \theta_{2k_0+1}$, $S_{nm} = \{z = p_n e^{i\theta}: \theta_{2m-1} + \eta' \leq \theta \leq \theta_{2m+1} + \eta'\}$, where p_n is taken from (2.13), while $\eta' = \eta/2(2k_0 + 1)$. On $S_{nm} \backslash E^{2m}$, by virtue of (2.10), (2.11), and (2.13), we have ($z = re^{i\theta} \in S_{nm} \backslash E^{2m}$) for $n \to \infty$

$$|H_{2m}(ze^{-i\theta_{2m}})| \geqslant \exp\{(S_\rho(\theta - \theta_{2m}) + o(1))\lambda(p_n)\},$$
$$|f_{2m}(z)| \leqslant \exp\{(h_{2m}(\theta) + o(1))\lambda(p_n)\},$$
$$|g_{2m}(z)| \leqslant \exp\{(h_{2m}(\theta) + o(1))\lambda(p_n)\},$$

where $h_{2m}(\theta) = \max\{S_\rho(\theta - \theta_{2k}): k \in \mathbf{Z} \backslash \{m\}\} \leq S_\rho(\theta - \theta_{2m}) - \kappa_m$, $\kappa_m > 0$. From here and from (2.22) it follows that on $\bigcup_{n=1}^{\infty}(S_{nm} \backslash E^{2m})$ the function f tends uniformly to a_{2m} for $r = |z| \to \infty$ and, consequently, $\ln^+|f(z) - a_j|^{-1} = O(1)$ for $r \to \infty$. Let

$$I_n = \{z : |z| = p_n\} \backslash \bigcup_{m=-k_0}^{k_0}(S_{nm} \backslash E^{2m}).$$

Then

$$m(p_n, a_j, f) = \frac{1}{2\pi}\int_{I_n} \ln^+ \frac{1}{|f(p_n e^{i\theta}) - a_j|}\, d\theta + O(1) =$$

$$= \frac{1}{2\pi}\int_{I_n} \ln^+|\psi_2(p_n e^{i\theta})|\, d\theta + \frac{1}{2\pi}\int_{I_n} \ln^+ \frac{1}{|f_j(p_n e^{i\theta}) - e^{i\alpha}|}\, d\theta + O(1) \leqslant \frac{3\eta}{2\pi}(1 + o(1))\lambda(p_n) + m(p_n, e^{i\alpha}, f_j) =$$

$$= \frac{3\eta}{2\pi}(1 + o(1))\lambda(p_n) + O(2^n) = \frac{3\eta}{2\pi}(1 + o(1))\lambda(p_n),\ n \to \infty,$$

by virtue of (2.9) and (2.20) since $\ln\lambda(p_n) = (1 + o(1))\rho \ln p_n \geq (1 + o(1))\rho \ln R_n = (1 + o(1))\rho e_2(n)$. From (2.27) and the arbitrariness of $\eta > 0$ it follows that $\delta(a_{2k+1}, f) = 0$.

It remains to show that $\beta(a, f) = 0$ for $a \notin E_2$. First we establish that $\delta(a, f) = 0$ for $a \notin E_2$. The segment $[-\pi + \eta, \pi - \eta]$, $\eta < 1/3$, contains a finite number of points θ_m'. We cover each of these points by an interval so that the total length of these intervals should not exceed η. The complement with respect to $[-\pi + \eta, \pi - \eta]$ of the union of these intervals consists of a finite number of segments Θ_j, $j_1 \leq j \leq j_2$, $\Theta_j \subset [\theta_{j-1}', \theta_j']$. Making use of (2.10), (2.14), and (2.22), we show, in the same way as above, that $f(q_n e^{i\theta})$ tends, uniformly with respect to $\theta \in \Theta_j$, to a_j when $n \to \infty$, $j_1 \leq j \leq j_2$. Therefore,

$$\ln^+|f(q_n e^{i\theta}) - a|^{-1} = O(1),\ n \to \infty,\ \theta \in \bigcup_{j=j_1}^{j_2}\Theta_j.$$

If

$$S = [-\pi, \pi] \backslash \bigcup_{j=j_1}^{j_2}\Theta_j,$$

then mes S ≤ 3η. Making use of the theorem of Edrei and Fuchs [1, p. 58], we obtain

$$m(q_n, a, f) = \frac{1}{2\pi}\int_S \ln^+ \frac{1}{|f(q_n e^{i\theta}) - a|} d\theta + O(1) \leqslant K_\mu \ln \frac{1}{3\eta} T(2q_n, f) + O(1), \ K = \text{const} > 0. \quad (2.28)$$

For $r \in [r_n', r_n''/2]$, by virtue of property 3 of Lemma 2.1, (2.21), and (2.27), for $r \to \infty$ we have

$$T(2r, f) \leqslant (2 + o(1))\lambda(2r) = (1 + o(1))2^{\rho+1}c_n r^\rho = \\ = (1 + o(1))2^{\rho+1}\lambda(r) \leqslant (1 + o(1))2^{\rho+1}L^{-1}T(r, f). \quad (2.29)$$

In particular, (2.29) is valid for $r = q_n$ and from (2.28) for $n \to \infty$ we obtain that

$$m(q_n, a, f) \leqslant (1 + o(1))K2^{\rho+1}L^{-1}\eta \ln \frac{1}{3\eta} T(q_n, f). \quad (2.30)$$

Taking into account that η can be selected arbitrarily small, we obtain $\delta(a, f) = 0$, $a \notin E_2$.

By virtue of (2.29) we have

$$T(2r, f) = O(T(r, f)), \ r \in \bigcup_{n=1}^{\infty} [r_n', r_n''/2], \ r \to \infty,$$

while by (2.30), $m(q_n, a, f) = o(T(q_n, f))$, $r_n' = o(q_n)$, $q_n = o(r_n'')$, $n \to \infty$. By a theorem of one of the authors [7], we have then $\beta(a, f) = 0$. Theorem 1 is entirely proved.

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