The relation GR = R allows us to lift the action of the group G by the automorphisms $\alpha(g)$ on (S, μ) to an action by the strict automorphisms $\bar{\alpha}(g)$ of the groupoid (R, ν): $\bar{\alpha}(g)(x, y) = (\alpha(g)x, \alpha(g)y)$ for $(x, y) \in R$. This gives us the possibility to construct the semidirect product $Gs_{\bar{\alpha}}R$, which we take as the definition of the semidirect product $Gs_{\bar{\alpha}}\Omega$.

LITERATURE CITED

1. J. Feldman, P. Hahn, and C. C. Moore, "Orbit structure and countable sections for actions of continuous groups," Adv. in Math., 28, 186-230 (1978).
2. A. Ramsay, "Virtual groups and group actions," Adv. in Math., 6, No. 3, 253-322 (1971).
3. J. Renault, A Groupoid Approach to C*-Algebras, Lecture Notes in Math., No. 793, Springer, Berlin (1980).
4. P. Hahn, "The regular representations of measure groupoids," Trans. Amer. Math. Soc., 242, 35-72 (1978).
5. J. Feldman and C. C. Moore, "Ergodic equivalence relation, cohomology, and von Neumann algebras. I, II," Trans. Amer. Math. Soc., 234, No. 2, 289-359 (1977).
6. G. W. Mackey, "Point realizations of transformation groups," Illinois J. Math., 6, Nos. 1-2, 327-335 (1962).

EXCEPTIONAL VALUES IN THE R. NEVANLINNA SENSE AND IN THE V. P. PETRENKO SENSE. 2

A. A. Gol'dberg, A. É. Eremenko, and M. L. Sodin

UDC 517.53

This paper is a continuation of [1]. In it we shall prove Theorems 2-4, formulated in the first part; we shall use the definitions and the notations in [1]. The numbering of the sections and formulas is continued.

3. By c_j we denote positive constants.

LEMMA 3.1. Let D be a Jordan domain, whose boundary contains the interval ℓ, and let v be a positive harmonic function in D, having zero limit values on ℓ. Then there exists a normal derivative $\partial v/\partial n$, continuous and positive on ℓ.

LEMMA 3.2. Let D be a domain containing the sector $\{z: |\arg z| < \pi/(2\alpha),\ |z| \leqslant R\}$, $\frac{1}{2} < \alpha < \infty$, $R > 0$, and assume that the segments $l_\pm = \{z = t\exp(\pm i\pi/(2\alpha)),\ 0 \leqslant t \leqslant R_0\}$, $R_0 < R$, lie on the boundary ∂D. Let v be a positive harmonic function in D, vanishing on $\ell_\pm$. Then there exist numbers $c_2 > c_1 > 0$ such that for $0 \leqslant r \leqslant R_0/2$ we have

$$c_1 r^\alpha \cos\alpha\theta \leqslant v(re^{i\theta}) \leqslant c_2 r^\alpha \cos\alpha\theta,\ |\theta| \leqslant \frac{\pi}{2\alpha}, \tag{3.1}$$

and, moreover, the left-hand side of this inequality is satified also for $r \leqslant R$.

LEMMA 3.3. Let D be a domain containing the sector $\left\{z: |\arg z| < \frac{\pi}{2\alpha},\ |z| \geqslant R\right\}$, $\frac{1}{2} < \alpha < \infty$, $R > 0$, and assume that the rays $l_\pm = \{z = t\exp(\pm i\pi/(2\alpha)),\ |z| \geqslant R_1\}$, $R_1 > R$, lie on the boundary ∂D. Let v be a positive harmonic function in D, vanishing on $\ell_\pm$ and unbounded in this sector. Then for $r \geqslant 2R_1$, $|\theta| \leqslant \pi/(2\alpha)$ the inequality (3.1) is satisfied and, moreover, the left-hand side of this inequality is satisfied also for $r \geqslant R$.

Statements of this type are well known; therefore, we shall omit of the proof of the lemmas. The following fact is also known (see, for example, [2], Sec. 2.3, Exercise 2).

LEMMA 3.4. Let D_1, D_2 be disjoint domains, $\partial D_1 \cap \partial D_2 = l$, where ℓ is a segment. Assume that the function v is harmonic in D_1, D_2, continuous in $D_1 \cup D_2 \cup l$ and equal to 0 on ℓ. By $\partial/\partial n_1$, $\partial/\partial n_2$ we denote the derivatives along the interior normals to the boundaries of the domains D_1, D_2, respectively. If $\left(\frac{\partial}{\partial n_1} + \frac{\partial}{\partial n_2}\right)v(z) \geqslant 0$, $z \in l$, then v is subharmonic in $D_1 \cup D_2 \cup l$.

Translated from Teoriya Funktsii, Funktsional'nyi Analiz i Ikh Prilozheniya, No. 48, pp. 58-70, 1987. Original article submitted October 16, 1985.

0090-4104/90/4902-0891$12.50

The following lemma plays a fundamental role at the proof of Theorem 3.

LEMMA 3.5. For any $\alpha\in\left(\frac{1}{2},\frac{6}{11}\right)$ there exists a function $u\in S$ with the following properties:

$$u(z)\equiv 0,\ |z|\leqslant 1; \tag{3.2}$$

$$|u(z)|\leqslant |z|^{\alpha},\ z\in \mathbf{C}; \tag{3.3}$$

$$\|u^{+}(re^{i\theta})\|\geqslant c_3 r^{\alpha},\ r\geqslant 4; \tag{3.4}$$

$$\|u^{-}(re^{i\theta})\|\geqslant \delta\|u^{+}(re^{i\theta})\|,\ r>0,\ \delta>0. \tag{3.5}$$

Proof. If $\beta=\alpha/(2\alpha-1)$, then $\pi/(2\beta)=\pi-\pi/(2\alpha)$, $\beta>6$. We set $D=\{z:|z|>2\}\cup\left\{z:|\arg(z-1)|<\frac{3\pi}{2\beta}\right\}$, $D_1=\left\{z:|\arg(z-1)|<\frac{\pi}{2\beta}\right\}$, $D_0=D\diagdown D_1$. Let $v_0=q\,\mathrm{Re}\,\Psi$, where Ψ is the function that maps D_0 conformally and univalently onto $\{w:\mathrm{Re}\,w>0\}$, $\Psi(\infty)=\infty$, $q>0$ (Fig. 1). Obviously, v_0 is a positive harmonic function in D_0, having on ∂D_0 zero limiting values. By virtue of Lemmas 3.2 and 3.3, the factor q can be selected so that

$$v_0(z)\leqslant |z|^{\alpha},\ z\in D_0; \tag{3.6}$$

$$v_0(z)\leqslant (|z|-1)^{\beta},\ z\in D_0,\ |z|\leqslant 6. \tag{3.7}$$

Further, by virtue of Lemma 3.2 we have

$$v_0(re^{i\theta}+1)\geqslant -c_4 r^{\beta}\cos\beta\theta,\ r\leqslant 6,\ \left|\theta\pm\frac{\pi}{\beta}\right|\leqslant\frac{\pi}{2\beta}, \tag{3.8}$$

while by virtue of Lemma 3.3 for $r\geqslant 4$, $|\theta|\in\left[\pi-\frac{\pi}{2\alpha},\pi\right]$ we have

$$c_5 r^{\alpha}\cos\alpha(\pi-|\theta|)\leqslant v_0(re^{i\theta}+1)\leqslant c_6 r^{\alpha}\cos\alpha(\pi-|\theta|). \tag{3.9}$$

We set

$$v(re^{i\theta}+1)=r^{\alpha}\cos\alpha(\pi-|\theta|),\ re^{i\theta}+1\in D_1,\ |\theta|<\pi \tag{3.10}$$

and then in the sector $D_1\cup\{z:|z-1|<5\}$ we replace this function by its smallest harmonic majorant v^* and we set

$$v^*(z)=v(z),\ z\in D_1\cap\{z:|z-1|\geqslant 5\}. \tag{3.11}$$

Then for $r\leqslant 5$, $|\theta|\leqslant\frac{\pi}{2\beta}$ we have

$$-c_7 r^{\beta}\cos\beta\theta\leqslant v^*(re^{i\theta}+1)\leqslant -c_8 r^{\beta}\cos\beta\theta. \tag{3.12}$$

Indeed, we consider in the sector $0<r<5$, $|\theta|<\pi/(2\beta)$ the harmonic function $V(re^{i\theta})=v^*(re^{i\theta}+1)+kr^{\beta}\cos\beta\theta$, $k>0$. This function vanishes on the boundary segments. By virtue of (3.10), selecting a sufficiently large k, the function V can be made nonnegative also on the arc $\left\{r=5,\ |\theta|\leqslant\frac{\pi}{2\beta}\right\}$. By the minimum principle we have $V(z)\geqslant 0$, which gives the left-hand side of the inequality (3.12). The right-hand side of this inequality is obtained in a similar manner.

We define a function u in the following manner:

$$u(z)=v_0(z),\ z\in\bar{D}_0; \tag{3.13}$$

$$u(z)=cv^*(z),\ z\in D_1,\ c>0; \tag{3.14}$$

$$u(z)\equiv 0,\ z\in\mathbf{C}\diagdown(\bar{D}_0\cup D_1). \tag{3.15}$$

We show that the constant c from (3.14) can be selected so that $u\in S$. Obviously, u is continuous in $\mathbf{C}$ and subharmonic outside the rays $l_{\pm}=\{z:\arg(z-1)=\pm\pi/(2\beta)\}$. By virtue of (3.9)-(3.11), (3.13), (3.14) and Lemmas 3.1 and 3.4, the function u is subharmonic in $\{z:|z-1|>5\}$, provided $c<c_5$. By virtue of (3.8), (3.12)-(3.14) and Lemmas 3.1 and 3.4, the function u is subharmonic in the neighborhoods $l_{\pm}\cap\{z:|z-1|<5\}$, provided $c<c_4/c_7$. At the three points $z=1$, $z=1+5\exp\left(\pm i\frac{\pi}{2\beta}\right)$ the subharmonicity is not violated since u(z) is continuous and subharmonic in the deleted neighborhoods of these points (see, for example, [3], Theorem 5.18).

Diminishing, if necessary, the constant c, we shall assume that

$$u(z)\geqslant -|z|^{\alpha},\ z\in\bar{D}_1. \tag{3.16}$$

In the sequel we shall need the following estimates:

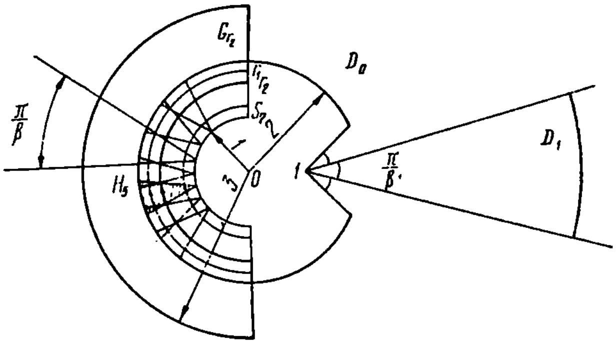

Fig. 1

$$u(re^{i\theta}+1)\leqslant -\varkappa_1 B(r+1,\ u),\ r>0,\ |\theta|\leqslant\frac{\pi}{4\beta},\ \varkappa_1>0; \tag{3.17}$$

$$|u(z)|\leqslant c_9(|z|-1)^\beta,\quad 1\leqslant|z|\leqslant 6. \tag{3.18}$$

The estimate (3.17) follows from (3.7), (3.9)-(3.15), while the estimate (3.18) from (3.7), (3.12)-(3.15).

We show that the function u satisfies (3.2), the estimate (3.3) from (3.6), (3.13), and (3.16), the estimate (3.4) from (3.9) and (3.13). For $1\leqslant r\leqslant 4$, by virtue of (3.14), (3.12), and (3.18) we have

$$\|u^-(re^{i\theta})\|=c\|v^*(re^{i\theta})\|\geqslant c_{10}(r-1)^\beta\geqslant c_{11}\|u^+(re^{i\theta})\|,$$

while for $r\geqslant 4$, by virtue of (3.14), (3.10), (3.11), and (3.3) we have

$$\|u^-(re^{i\theta})\|=c\|v^*(re^{i\theta})\|\geqslant c_{12}r^\alpha\geqslant c_{12}\|u^-(re^{i\theta})\|,$$

thus, inequality (3.5) is proved. The lemma is proved.

For the proof of Theorem 2 we have to modify somewhat the function u, replacing the inequality (3.5) by the "opposite" inequality (3.23).

LEMMA 3. For each $\alpha\in\left(\frac{1}{2},\ \frac{6}{11}\right)$ there exist a constant $\varkappa>0$ and sequences $(u_n)\ u_n\in S$, and $(s_n),\ 1<s_n<2,\ n\in N$, such that the functions u_n satisfy the following conditions:

$$u_n(z)\equiv 0,\quad |z|\leqslant 1; \tag{3.19}$$
$$|u_n(z)|\leqslant c_{13}|z|^\alpha,\quad z\in C; \tag{3.20}$$
$$|u_n(z)|\leqslant c_{14}(|z|-1)^\beta,\quad 1\leqslant|z|\leqslant 6,\quad \beta=\alpha/(2\alpha-1); \tag{3.21}$$
$$B(r,\ u_n)\geqslant\|u_n^+(re^{i\theta})\|\geqslant c_{15}r^\alpha,\quad r\geqslant 4; \tag{3.22}$$
$$\|u_n^-(s_ne^{i\theta})\|\leqslant\frac{1}{n}\|u_n^+(s_ne^{i\theta})\|; \tag{3.23}$$
$$u_n(re^{i\theta}+1)\leqslant -\varkappa B(r+1,\ u_n),\quad r>0,\quad |\theta|\leqslant\pi/(4\beta),\quad \varkappa>0. \tag{3.24}$$

Proof. We shall use the domains and the functions constructed at the proof of Lemma 3.5.

For each t, 1 < t < 2, let $G_t=\{z:\operatorname{Re}z<0,\ t<|z|<3\}$. In the semiannulus G_t we replace the function u by its smallest harmonic majorant. We denote the obtained function by w_t. It is easy to see that

$$w_t\rightrightarrows w_1,\quad t\to 1, \tag{3.25}$$

uniformly in C.

We introduce the sectors $H_k=\left\{z:z\exp\left(-i\pi\left(\frac{3}{4}+\frac{k}{2\pi}\right)\right)\in D_1,\ |z|<3\right\}$, k = 0, 1,...,n, and the functions

$$v_k(z)=\begin{cases}-u\left(z\exp\left(-i\pi\left(\frac{3}{4}+\frac{k}{2\pi}\right)\right)\right), & z\in H_k,\\ 0,\ z\notin H_k, & |z|<3.\end{cases} \tag{3.26}$$

These are positive harmonic functions in H_k, subharmonic for |z| < 3. By virtue of (3.12) and (3.14) we have

$$0\leqslant v_k(z)\leqslant c_{16}(|z|-1)^\beta,\quad z\in G_1. \tag{3.27}$$

On the other hand,

$$w_1(z) \geqslant c_{17}(|z|-1),\ z\in G_1 \cap \{z:|\pi-\arg z|\leqslant 0{,}4\pi\},$$

in order to see this it is sufficient to apply to w_1 a modification of Lemma 3.1, in which D contains on its boundary an arc of a circumference instead of the interval ℓ. Therefore, there exists a number r_1, $1 < r_1 < 2$, such that $w_1(z) > 2v_k(z)$, $|z|=r_1$, $z\in G_1$, $k=0, 1, \ldots, n$. Now from (3.25) we conclude that there exists a number $r_2 < r_1$, sufficiently close to 1, such that

$$\{w_{r_2}(r_1e^{i\theta}): 0{,}6\pi \leqslant \theta \leqslant 1{,}4\pi\} > B(r_1, v_0) = \ldots = B(r_1, v_n). \tag{3.28}$$

We note that the numbers r_1, r_2 do not depend on n.

Now we define a function u_n:

$$u_n(z) = \begin{cases} w_{r_2}(z), & |z| > r_1; \\ \max\{w_{r_2}, v_0, \ldots, v_n\}, & |z| \leqslant r_1. \end{cases} \tag{3.29}$$

This function is subharmonic for $|z| > r_1$ since $w_{r_2}\in S$, while for $|z| < r_1$ as the upper envelope of a finite family of subharmonic functions. Finally, u_n is subharmonic in the neighborhood of the neighborhood of the circumference $\{z:|z|=r_1\}$ since, by virtue of (3.28) and (3.29), in some neighborhood of this circumference it coincides with w_{r_2}. Thus, $u_n\in S$. We mention at once that for $\mathrm{Re}\, z \geqslant 0$ or $|z|\geqslant 3$ we have

$$u_n(z) \equiv u(z). \tag{3.30}$$

We show that for the functions u_n the relations (3.19)-(3.24) are satisfied. Relation (3.19) is satisfied by construction. By virtue of (3.30) and (3.3), inequality (3.20) is satisfied for $|z|\geqslant 3$, while, by virtue of (3.19), also for $|z| < 3$ (not necessarily with the same value of the constant c_{13}). The inequality (3.21) for $1\leqslant|z|\leqslant r_2$ and for $3\leqslant|z|\leqslant 6$ follows from (3.18), (3.27), (3.29), and (3.30); consequently, it holds also for c $c_{15}=c_3$ The estimate (3.22) with $c_{15} = c_3$ follows directly from (3.4) and (3.30).

We select the number $s_n < r_2$ sufficiently close to unity in order that the sets $H_k \cap \{z: |z|=s_n\}$ be pairwise disjoint. Then by virtue of (3.30), (3.29), and (3.26) we have $u_n(z)\geqslant 0$ for Re z < 0 and

$$\|u_n^-(s_ne^{i\theta})\| = \|u^-(s_ne^{i\theta})\| \leqslant \frac{1}{n}((n+1)\|u^-(s_ne^{i\theta})\| + \|u^+(s_ne^{i\theta})\|) = \frac{1}{n}\|u_n^+(s_ne^{i\theta})\|,$$

which yields (3.23).

Finally, by virtue of (3.30), (3.17), (3.8), (3.26), (3.21), and (3.29), for $|\theta|\leqslant\pi/(4\beta)$ u_n we have $(1+re^{i\theta}) = u(1+re^{i\theta}) \leqslant -\varkappa_1 B(1+r, u) \leqslant -\varkappa_1 c_{18}\times B(1+r, u_n)$, which gives (3.24) with $\varkappa=\varkappa_1 c_{18}$. Lemma 3.6 is proved.

Proof of Theorem 2. First we construct a function $U\in S$ of order ρ, $\frac{1}{2}<\rho\leqslant 1$, such that

$$\varliminf_{r\to\infty} \|U^-(re^{i\theta})\|/\|U^+(re^{i\theta})\| = 0, \tag{3.31}$$

and at the same time

$$\varliminf_{r\to\infty} \min\{U^-(z):|z|=r,\ z\in K\}/B(r, U) > 0, \tag{3.32}$$

where
$$K = \{z: \mathrm{Re}\, z > 0,\ |\mathrm{Im}\, z| < \ln|z|,\ |z| > 1\}.$$

If $\rho < 6/11$, then we set $\alpha = \rho$; if, however, $\rho\geqslant 6/11$, then for α we take an arbitrary number $1/2 < \alpha < 6/11$. Then, according to Lemma 3.6, there exist functions u_n and numbers s_n, for which one has (3.19)-(3.24). We set $p_n=\|u_n^+(s_ne^{i\theta})\|$.

We construct inductively sequences of positive, unboundedly increasing numbers (A_k), (T_k). We set $A_1 = T_1 = 1$. If the numbers $A_1,\ldots,A_{n-1}$, $T_1,\ldots,T_{n-1}$ have been already selected, then we select T_n, $n\geqslant 2$, by observing the following conditions:

$$\ln T_n \geqslant n\sum_{k=1}^{n-1} A_k; \tag{3.33}$$

$$\ln T_n \geqslant A_{n-1}/p_n; \tag{3.34}$$

now we set

$$A_n = T_n^{\rho}\ln T_n, \quad n\geqslant 2. \tag{3.35}$$

We set $U_k(z)=A_k u_k(z/T_k)$, $U(z)=\sum_{k=1}^{\infty} U_k(z)$. By virtue of (3.19), this series converges uniformly on each compact and, consequently, $U\in S$.

Assume further that $T_n\leqslant|z|=r<T_{n+1}$. Then

$$U(z)=U_{n-1}(z)+U_n(z)+o(1)\,\|U_{n-1}^{+}(re^{i\theta})\|,\quad n\to\infty, \tag{3.36}$$

uniformly with respect to z. Indeed, by virtue of (3.19), (3.20), (3.33), (3.35), and (3.22) we have

$$|U(z)-U_{n-1}(z)-U_n(z)|\leqslant\sum_{k=1}^{n-2}A_k|u_k(z/T_k)|\leqslant c_{13}r^{\alpha}\sum_{k=1}^{n-2}A_k\leqslant$$
$$\leqslant\frac{c_{13}}{n}r^{\alpha}\ln T_{n-1}\leqslant\frac{c_{13}}{n}A_{n-1}\frac{r^{\alpha}}{T_{n-1}^{\rho}}\leqslant\frac{c_{13}}{n}A_{n-1}\frac{r^{\alpha}}{T_{n-1}^{\alpha}}\leqslant c_{19}n^{-1}\|U_{n-1}^{+}(re^{i\theta})\|.$$

If, however, $T_ns_n\leqslant r=|z|<T_{n+1}$, then

$$U(z)=U_n(z)+o(1)\,\|U_n^{+}(re^{i\theta})\|,\quad n\to\infty, \tag{3.37}$$

again uniformly with respect to z. Indeed, assume first that $T_ns_n\leqslant r\leqslant 5T_n$. Then, by virtue of (3.20), (3.34), and (3.35), we have

$$|U_{n-1}(z)|\leqslant c_{13}A_{n-1}\frac{r^{\alpha}}{T_{n-1}^{\alpha}}\leqslant c_{20}A_{n-1}\frac{T_n^{\alpha}}{T_{n-1}^{\alpha}}\leqslant c_{20}p_n\frac{T_n^{\alpha}}{T_{n-1}^{\alpha}}\ln T_n\leqslant\frac{c_{20}}{T_{n-1}^{\alpha}}A_np_n=\frac{c_{20}}{T_{n-1}^{\alpha}}\|U_n^{+}(T_ns_ne^{i\theta})\|,$$

which, together with (3.36), yields (3.37). If, however, $5T_n\leqslant r\leqslant T_{n+1}$, then, making use of (3.20), (3.33), (3.35), (3.22), we obtain

$$|U_{n-1}(z)|\leqslant c_{13}A_{n-1}\frac{r^{\alpha}}{T_{n-1}^{\alpha}}\leqslant\frac{c_{13}}{n}r^{\alpha}\ln T_n=\frac{c_{13}}{n}\frac{A_nr^{\alpha}}{T_n^{\rho}}\leqslant\frac{c_{13}}{n}A_n\frac{r^{\alpha}}{T_n^{\alpha}}\leqslant\frac{c_{21}}{n}\|U_n^{+}(re^{i\theta})\|,$$

which, together with (3.36), yields again (3.37).

Now we find the order of the function U. By virtue of (3.36) and (3.35), for $T_n\leqslant r<T_{n+1}$ we have $B(r,U)\leqslant(1+o(1))B(r,U_{n-1})+B(r,U_n)\leqslant(1+o(1))c_{13}\frac{A_{n-1}}{T_{n-1}^{\alpha}}r^{\alpha}+c_{13}\frac{A_n}{T_n^{\alpha}}r^{\alpha}\leqslant(1+o(1))c_{13}r^{\alpha}T_{n-1}^{\rho-\alpha}\times\ln T_{n-1}+c_{13}r^{\alpha}T_n^{\rho-\alpha}\ln T_n\leqslant c_{22}r^{\alpha}T_n^{\rho-\alpha}\ln T_n\leqslant c_{22}r^{\rho}\ln r$. On the other hand, by virtue of (3.37), (3.34), (3.35), for $n\to\infty$ we have $B(T_ns_n,U)\geqslant(1+o(1))\|U_n^{+}(T_ns_ne^{i\theta})\|=(1+o(1))A_np_n\geqslant(1+o(1))\frac{A_{n-1}}{\ln T_n}T_n^{\rho}\ln T_n\geqslant(1+o(1))T_n^{\rho}\geqslant(2^{-\rho}+o(1))(T_ns_n)^{\rho}$. Therefore, the order of the function U is equal to ρ.

Further, making use of (3.37) and (3.23), for $n\to\infty$ we obtain

$$\|U^{-}(T_ns_ne^{i\theta})\|\leqslant\|U_n^{-}(T_ns_ne^{i\theta})\|+o(1)\|U_n^{+}(T_ns_ne^{i\theta})\|\leqslant\left(\frac{1}{n}+o(1)\right)\|U_n^{+}(T_ns_ne^{i\theta})\|=o(1)\|U^{+}(T_ns_ne^{i\theta})\|, \tag{3.38}$$

which gives (3.31).

Now we prove the estimate (3.32). Assume first that $T_n\leqslant r=|z|\leqslant T_n+\ln^3T_n$. Then, by virtue of (3.21), (3.35), we have

$$|U_n(z)|\leqslant c_{14}A_n\left(\frac{|z|}{T_n}-1\right)^{\beta}\leqslant c_{14}A_n\frac{\ln^{3\beta}T_n}{T_n^{\beta}}=c_{14}T_n^{\rho-\beta}\ln^{3\beta+1}T_n=o(1),\quad n\to\infty, \tag{3.39}$$

since $\rho\leqslant1<6<\beta$. Therefore, by virtue of (3.36), (3.20), (3.22), and (3.24), we have

$$U(z)=U_{n-1}(z)+o(1)B(r,U_{n-1})\leqslant-(\varkappa+o(1))B(r,U),$$

if $z\in K$, i.e., condition (3.22) holds. Assume now that $T_n+\ln^3T_n\leqslant r=|z|<T_{n+1}$, $z\in K$. In this case, in view of (3.26), (3.20), (3.22), (3.24), for $n\to\infty$ we have: $U(z)=U_{n-1}(z)+U_n(z)+o(1)\|U_{n-1}^{+}(re^{i\theta})\|\leqslant(-\varkappa+o(1))B(r,U_{n-1})-\varkappa B(T_n+|z-T_n|,U_n)\leqslant(-\varkappa+o(1))B(r,U_{n-1})-\varkappa B(r,U_n)\leqslant(-\varkappa+o(1))B(r,U)$. Thus, inequality (3.22) is proved.

Making use of Theorem Y, we approximate the function $U\in S$ by the logarithm of the modulus of the entire function f so that (2.17) be satisfied. It is easy to see that the order of the entire function f is also equal to ρ. Since $U_n(-r)\geqslant0$, $r\geqslant0$, $n\in N$, we have $B(r,U)\geqslant U(-r)\geqslant U_1(-r)=u_1(-r)\geqslant c_5(r+1)^{\alpha}$ for $r\geqslant4$ by virtue of (3.9), (3.13), (3.30).

By virtue of (3.36), (3.19), (3.20), (3.35), and (3.33) we have $B(T_n, U) = (1+o(1))B(T_n, U_{n-1}) \leqslant (c_{13}+o(1))A_{n-1}\frac{T_n^\alpha}{T_{n-1}^\alpha} \leqslant (c_{13}+o(1))T_n^\alpha \ln T_n$, $n\to\infty$, and, therefore, the lower order of the function U is equal to α.

Now we note that

$$\| U(re^{i\theta}) - \ln|f(re^{i\theta})|\| = o(r^\alpha), \quad r\to\infty. \tag{3.40}$$

Indeed, let E(r) be the intersection of the exceptional set from Theorem Y with the circumference $\{z:|z|=z\}$. Then, by virtue of (2.17) and by the theorem of Edrei and Fuchs on small arcs [4, p. 58], already used in Sec. 2, applied both to U and to f, we have

$$\| U(re^{i\theta}) - \ln|f(re^{i\theta})|\| = \frac{1}{2\pi}\left\{ \int\limits_{[0,2\pi)\setminus E(r)} + \int\limits_{E(r)} \right\} |U(re^{i\theta}) - $$
$$- \ln|f(re^{i\theta})||\,d\theta = O(\ln^2 r) + O(r^{\rho-1}\ln r) = o(r^\alpha), \quad r\to\infty.$$

Now from (3.38) and (3.40) we obtain

$$m(T_n s_n, 0, f) \leqslant \|U^-(T_n s_n e^{i\theta}) - \ln^-|f(T_n s_n e^{i\theta})|\| + \|U^-(T_n s_n e^{i\theta})\| =$$
$$= o(T_n^\alpha) + o(1)\|U^+(T_n s_n e^{i\theta})\| \leqslant o(T_n^\alpha) + o(1)\|U^+(T_n s_n e^{i\theta}) -$$
$$- \ln^+|f(T_n s_n e^{i\theta})|\| + o(1)\|\ln^+|f(T_n s_n e^{i\theta})|\| = o(T_n^\alpha) +$$
$$+ o(1)T(T_n s_n, f) = o(1)T(T_n s_n, f), \quad n\to\infty,$$

which gives δ(0, f) = 0.

Since $E\in(\sigma)$, for sufficiently large r the set E cannot cover entirely the arc $\{z:|z|=r\}\cap K$. Therefore, from (3.32) there follows that β(0, r) > 0.

The unknown function has been constructed for all ρ, $1/2<\rho\leqslant 1$. Considering the function $g(z)=f(z^n)$, $n\in N$, we can obtain any preassigned order ρ > 1/2. Theorem 2 is proved.

Remark 3.1. For ρ < 6/11 the lower order of the constructed function f coincides with ρ. For $6/11\leqslant\rho\leqslant 1$ this can be achieved, complicating the construction of the auxiliary functions u_n from Lemma 3.6.

Remark 3.2. For the constructed function f we have

$$\varliminf_{r\to\infty} \ln^+|f(r)|/\ln M(r, f) > 0. \tag{3.41}$$

Indeed, let

$$E\subset \bigcup_{k=1}^{\infty}\{z:|z-z_k|<r_k\}, \quad \sum_{k=1}^{\infty} r_k \leqslant M, \tag{3.42}$$

where E is the exceptional set from Theorem Y (see [1]). We note that in the circle $\{z:|z-r|\leqslant 2M+1\}$ there exists the circumference $\{z:|z-r|=t\}$, not intersecting E. Now, by virtue of the mean value theorem, Theorem Y, (3.36), and (3.32), for n → ∞ we have

$$\ln|f(r)| \leqslant \frac{1}{2\pi}\int_{-\pi}^{\pi}\ln|f(r+te^{i\psi})|\,d\psi \leqslant \frac{1}{2\pi}\int_{-\pi}^{\pi}U(r+te^{i\psi})\,d\psi +$$
$$+ o(r^\alpha) \leqslant \frac{1}{2\pi}\int_{-\pi}^{\pi}U_{n-1}(r+te^{i\psi})\,d\psi + \frac{1}{2\pi}\int_{-\pi}^{\pi}U_n(r+te^{i\psi})\,d\psi + o(1)B(r, U_{n-1}), \tag{3.43}$$

if $T_n\leqslant r\leqslant T_{n+1}$. If now $T_n\leqslant r\leqslant T_n+\frac{1}{2}\ln^3 T_n$, then for $n\geqslant n_0$ we have $r+t\leqslant r+2M+1\leqslant T_n+\ln^3 T_n$ and, by virtue of (3.43), (3.39), (3.24), (3.20), and (3.22), we have

$$\ln|f(r)| \leqslant \frac{1}{2\pi}\int_{-\pi}^{\pi}U_{n-1}(r+te^{i\psi})\,d\psi + o(1)B(r, U_{n-1}) \leqslant$$
$$\leqslant -\varkappa B(T_{n-1}+|r-T_{n-1}+te^{i\psi}|, U_{n-1}) + o(1)B(r, U_{n-1}) \leqslant$$
$$\leqslant -\varkappa B(r-2M-1, U_{n-1}) + o(1)B(r, U_{n-1}) \leqslant (-c_{24}+o(1))B(r, U_{n-1}) = (-c_{24}+o(1))B(r, U). \tag{3.44}$$

Assume now that $T_n+\frac{1}{2}\ln^3 T_n\leqslant r\leqslant T_{n+1}$. Then for $|\psi|\leqslant\pi/2$ we have $r\leqslant|r+te^{i\psi}|\leqslant T_n+|r-T_n+te^{i\psi}|$. By virtue of (3.43), (3.24), (3.36), inequality $U_n(r+te^{i\psi})\leqslant 0$ and of the already given estimates for n → ∞, we have

$$\ln|f(r)| \leqslant \frac{1}{2\pi}\int_{-\pi}^{\pi} U_{n-1}(r+te^{i\psi})\,d\psi + \frac{1}{2\pi}\int_{-\pi/2}^{\pi/2} U_n(r+te^{i\psi})\,d\psi +$$
$$+ o(1)\,B(r, U_{n-1}) \leqslant (-c_{24}+o(1))\,B(r, U_{n-1}) -$$
$$-\frac{\varkappa}{2}B(T_n+|r-T_n+te^{i\psi}|, U_n) \leqslant (-c_{24}+o(1))\,B(r, U_{n-1}) - \frac{\varkappa}{2}B(r, U_n) \leqslant (-c_{25}+o(1))\,B(r, U). \quad (3.45)$$

By virtue of Lemma 2.2 and Theorem Y from [1], we have

$$\ln M(r, f) \sim B(r, U),\ r\to\infty. \quad (3.46)$$

Now (3.41) follows from (3.44)-(3.46).

The proof of Theorem 3 repeats entirely the proof of Theorem 2 with the only difference that the function u_n from Lemma 3.6 must be replaced by the function u from Lemma 3.5, inequality (3.22) is not needed, while the estimate (3.31) has to be replaced by the inequality $\|U^-(re^{i\theta})\| \geqslant (\delta+o(1))\|U^+(re^{i\theta})\|, r\to\infty, \delta>0$, which holds by virtue of (3.5). By virtue of this inequality, for the entire function f, occurring in (3.40), we have $\delta(0, f) > 0$. Further, from (3.2), (3.3), (3.35), and (3.36) there follows that

$$B(T_n, U) = (1+o(1))\,B(T_n, U_{n-1}) \leqslant \frac{1+o(1)}{T_{n-1}^{\alpha}}T_n^{\alpha}\ln T_n,\quad n\to\infty.$$

Now by virtue of (3.37), (3.4), and (3.35), we have $\|U^+(4T_ne^{i\theta})\| = (1+o(1))\|U_n^+(4T_ne^{i\theta})\| \geqslant (c_{26}+o(1))\,A_n = (c_{26}+o(1))\,T_n^{\rho}\ln T_n \geqslant (c_{26}+o(1))\,T_{n-1}^{\alpha}B(T_n, U)$. From here and from (3.40) we conclude that the function f satisfies all the requirements of Theorem 3.

4. Proof of Theorem 4. Let $2<\mu<\infty$, $\eta=\frac{1}{2}\min\left(\frac{\pi}{\mu}, \frac{\pi}{2}-\frac{\pi}{\mu}\right)$. We consider the domain

$$D_1 = \left\{z = 1+re^{i\theta} : r>0,\ |\theta| < \eta+\frac{\pi}{\mu}\right\},$$
$$D_2 = \left\{z = -1+re^{i\theta} : r>0,\ |\theta-\pi| < \frac{\pi}{2\mu}\right\}.$$

We set

$$w_1(z) = \begin{cases} r^{\mu}\sin\mu(|\theta|-\eta), & z\in D_1;\\ 0, & z\in C\setminus D_1;\end{cases}$$
$$w_2(z) = \begin{cases} r^{\mu}\cos\mu(\theta-\pi), & z\in D_2,\\ 0, & z\in C\setminus D_2.\end{cases}$$

We consider the function $w = w_1 + w_2$, defined in C. Clearly, $w\in S$. The constructed function has the following properties:

$$w(z) = 0,\quad |z|\leqslant 1; \quad (4.1)$$
$$|w(z)| \leqslant |z|^{\mu},\quad z\in C; \quad (4.2)$$
$$\|w^-(re^{i\theta})\| \geqslant \delta\|w^+(re^{i\theta})\|,\quad r\geqslant 1, \quad (4.3)$$

where the positive constant δ depends only on μ. Further,

$$w(-t+e^{i\theta}) \geqslant 0,\quad 0\leqslant\theta\leqslant 2\pi,\quad t>0; \quad (4.4)$$
$$\|w^+(-t+e^{i\theta})\| \geqslant \alpha t^{\mu+1},\quad 0\leqslant t\leqslant 1; \quad (4.5)$$
$$\|w^+(re^{i\theta})\| \geqslant \beta r^{\mu},\quad r\geqslant 2; \quad (4.6)$$
$$\|w^+(re^{i\theta})\| \geqslant \gamma(r-1)^{\mu+1},\quad 1\leqslant r\leqslant 2, \quad (4.7)$$

where the positive numbers α, β, γ depend only on μ. Let $R_k\geqslant 2$, $R_k\uparrow\infty$, $a_k>0$, $u_k(z) = a_kw(z/R_k)$. By virtue of (4.1) the series

$$u(z) = \sum_{k=1}^{\infty} u_k(z)$$

converges uniformly on the compacta in C and $u(z)\in S$.

We see $a_k = R_k^{2\mu+1}\ln^2 R_k$, and we select the sequence (R_k) so that for $k\geqslant 2$ we should have

$$\ln R_k > \max\{2^k, R_{k-1}^{2\mu+1}\ln^2 R_{k-1}\} = \max\{2^k, a_{k-1}\}. \quad (4.8)$$

We estimate the order of the function u. If $|z|\geqslant R_k$, then by virtue of (4.2) and (4.8) we have

$$u_k(z) \leqslant R_k^{2\mu+1} \ln^2 R_k \frac{|z|^\mu}{R_k^\mu} \leqslant |z|^{2\mu+1} \ln^2 |z| \leqslant 2^{-k} |z|^{2\mu+1} (\ln^+ |z|)^3.$$

If $|z| < R_k$, then, in view of (4.1), the last inequality is obviously satisfied. Thus, the order of the function u does not exceed $2\mu + 1$. Since all $u_k(-r) \geqslant 0$ for $r > 0$, we have

$$u(-2R_k) \geqslant u_k(-2R_k) = R_k^{2\mu+1} (\ln^2 R_k) w(-2) = R_k^{2\mu+1} (\ln^2 R_k).$$

Consequently, the order of the function u is equal to $2\mu + 1$.

We show that the function u has a positive deficiency. Let $r \geqslant R_1$. We select the number k so that $R_k \leqslant r < R_{k+1}$.

Assume first that $2R_k \leqslant r < R_{k+1}$. By virtue of (4.6) we have

$$\| u_k^+ (re^{i\theta}) \| = R_k^{2\mu+1} \ln^2 R_k \left\| w^+ \left(\frac{r}{R_k} e^{i\theta} \right) \right\| \geqslant \beta R_k^{\mu+1} (\ln^2 R_k) r^\mu. \tag{4.9}$$

On the other hand, $u_m(re^{i\theta}) = 0$ for $m > k$ by virtue of (4.1) and, in addition, by virtue of (4.2), (4.8), and (4.9), we have

$$\left|\sum_{j=1}^{k-1} u_j (re^{i\theta})\right| \leqslant (k-1) R_{k-1}^{\mu+1} \ln^2 R_{k-1} r^\mu = (k-1) R_{k-1}^{\mu+1} (\ln^2 R_{k-1}) r^\mu = o(\| u_k^+ (re^{i\theta}) \|), \quad k \to \infty, \tag{4.10}$$

uniformly with respect to $r \in [2R_k, R_{k+1}]$. Consequently, for such r by virtue of (4.3) we have

$$\| u^- (re^{i\theta}) \| \geqslant (\delta + o(1)) \| u^+ (re^{i\theta}) \|, \quad r \to \infty. \tag{4.11}$$

Assume now that $R_k + 1 \leqslant r < 2R_k$. By virtue of (4.7) we have

$$\| u_k^+ (re^{i\theta}) \| = R_k^{2\mu+1} \ln^2 R_k \left\| w^+ \left(\frac{r}{R_k} e^{i\theta} \right) \right\| \geqslant$$

$$\geqslant R_k^{2\mu+1} \ln^2 R_k \| w^+ ((1 + R_k^{-1}) e^{i\theta}) \| \geqslant \gamma R_k^{2\mu+1} \ln^2 R_k R_k^{-\mu-1} = \gamma R_k^\mu \ln^2 R_k.$$

On the other hand, for $r < 2R_k$, by virtue of (4.2), (4.8) and the previous estimate we have

$$\left| \sum_{j=1}^{k-1} u_j (re^{i\theta}) \right| \leqslant (k-1) R_{k-1}^{2\mu+1} (\ln^2 R_{k-1}) \left(\frac{2R_k}{R_{k-1}} \right)^\mu = 2^\mu (k-1) R_{k-1}^{\mu+1} (\ln^2 R_{k-1}) R_k^\mu = o(1) \| u_k^+ (re^{i\theta}) \|, \quad k \to \infty,$$

uniformly with respect to $r \in [R_k + 1, 2R_k]$. Thus, by virtue of (4.3), the estimate (4.11) holds also in this case.

We consider the remaining case $R_k \leqslant r < R_k + 1$. From (4.10), replacing $k - 1$ by $k - 2$, there follows that $\left| \sum_{j=1}^{k-2} u_j (re^{i\theta}) \right\| = o(\| u_{k-1}^+ (re^{i\theta}) \|)$, $k \to \infty$.

We assume that $\| u_k^+ (re^{i\theta}) \| \geqslant \| u_{k-1}^+ (re^{i\theta}) \|$. Then

$$\| u^+ (re^{i\theta}) \| \leqslant (1 + o(1)) \| u_{k-1}^+ (re^{i\theta}) \| + \| u_k^+ (re^{i\theta}) \| \leqslant (2 + o(1)) \| u_k^+ (re^{i\theta}) \|, \quad r \to \infty. \tag{4.12}$$

We note that if $u_k(z) < 0$, then also $u_j(z) < 0$ for $1 \leqslant j \leqslant k - 1$. From here and from (4.1), (4.3), (4.12) there follows that

$$\| u^- (re^{i\theta}) \| \geqslant \| u_k^- (re^{i\theta}) \| \geq \delta \| u_k^+ (re^{i\theta}) \| \geqslant \left(\frac{\delta}{2} + o(1) \right) \| u^+ (re^{i\theta}) \|, \quad r \to \infty.$$

If , however, $\| u_k^+ (re^{i\theta}) \| < \| u_{k-1}^+ (re^{i\theta}) \|$, then, in the same way as (4.12), we obtain $\| u^+ (re^{i\theta}) \| \leqslant (2 + o(1)) \| u_{k-1}^+ (re^{i\theta}) \|$, $r \to \infty$. The sufficiently small number $u_k (re^{i\theta}) = 0$, $r \in [R_k, R_k + 1]$, $|\theta| \in \left[\varepsilon, \frac{\pi}{2}\right]$;

$$I(r) = \frac{1}{2\pi} \int_{|\theta| \geqslant \varepsilon} u_{k-1}^- (re^{i\theta}) \, d\theta \geqslant \frac{1}{2} \| u_{k-1}^- (re^{i\theta}) \|, \quad r \in [R_k, R_k + 1].$$

Then for $k \to \infty$ we have

$$\| u^- (re^{i\theta}) \| \geqslant o(\| u_{k-1}^+ (re^{i\theta}) \|) + \| (u_{k-1} (re^{i\theta}) + u_k (re^{i\theta}))^- \| \geqslant o(\| u_{k-1}^+(re^{i\theta}) \|) + I(r) \geqslant o(\| u_{k-1}^+(re^{i\theta}) \|) + \frac{1}{2} \times$$

$$\times \| u_{k-1}^- (re^{i\theta}) \| \geqslant \left(\frac{\delta}{2} + o(1) \right) \| u_{k-1}^+ (re^{i\theta}) \| \geqslant \left(\frac{\delta}{4} + o(1) \right) \| u^+ (re^{i\theta}) \|. \tag{4.13}$$

Thus, from (4.11)-(4.13) there follows that the function u has a positive deficiency.

We show that the deficiency of the function u(z − 1) is equal to 0. We have

$$\left|\sum_{j=1}^{k-1} u_j (R_k e^{i\theta} - 1)\right| \leqslant (k-1) R_{k-1}^{2\mu+1} \ln^2 R_{k-1} 2^\mu \frac{R_k^\mu}{R_{k-1}^\mu} = 2^\mu (k-1) R_{k-1}^{\mu+1} \ln^2 R_{k-1} R_k^\mu. \tag{4.14}$$

Further, by virtue of (4.5), we have $\|u_k^+ (R_k e^{i\theta} - 1)\| = R_k^{2\mu+1} \ln^2 R_k \left\| w^+ \left(e^{i\theta} - \frac{1}{R_k}\right)\right\| \geqslant \alpha R_k^{2\mu+1} \ln^2 R_k R_k^{-\mu-1} = \alpha R_k^\mu \ln^2 R_k$. Therefore, from (4.14), (4.1), and (4.8) we conclude that for k → ∞ we have $|u(R_k e^{i\theta} - 1) - u_k (R_k e^{i\theta} - 1)| = o(\|u_k^+ (R_k e^{i\theta} - 1)\|)$. But by virtue of (4.4), $u_k (R_k e^{i\theta} - 1) \geqslant 0$, and, therefore, for k → ∞ we have $\|u^- (R_k e^{i\theta} - 1)\| = o(\|u_k^+ (R_k e^{i\theta} - 1)\|) = o(\|u^+ (R_k e^{i\theta} - 1)\|)$; this is what we intended to prove.

Modifying some of the steps of the proof of R. S. Yulmukhametov's theorem [5], we can obtain the following theorem.*

<u>THEOREM Y'.</u> For each subharmonic function v(z) of finite order there exists an entire function f, satisfying the asymptotic inequality:

$$\|v(re^{i\theta} - a) - \ln|f(re^{i\theta} - a)|\| = O(\ln^2 r), \quad r \to \infty$$

for each $a \in \mathbf{C}$.

Applying this theorem to the constructed subharmonic function u(z), we obtain an entire function f for which the assertion of Theorem 4 holds.

<u>Remark 4.1.</u> For r > 0 we have $w(-r) \geqslant 0$. Therefore, for $r \geqslant 2R_1$ we have

$$u(-r) \geqslant u_1(-r) \geqslant \varkappa r^\mu, \quad \varkappa > 0.$$

Consequently, the lower order of the function u, and thus, also of f, is not smaller than $\mu = (\rho - 1)/2$.

The fundamental results of the present paper, Theorems 1 and 2 have been communicated by the authors in Dokl. Akad. Nauk Ukr. SSR, No. 10, 1984, pp. 3-5.

LITERATURE CITED

1. A. A. Gol'dberg, A. E. Eremenko, and M. L. Sodin, "Exceptional values in the R. Nevanlinna sense and in the V. P. Petrenko sense. 1," Teor. Funktsii Funktsional. Anal. i Prilozhen. (Kharkov), No. 47, 41-51 (1987).
2. M. A. Evgrafov, Asymptotic Estimates and Entire Functions [in Russian], Nauka, Moscow (1979).
3. W. K. Hayman and P. B. Kennedy, Subharmonic Functions, Academic Press, London (1976).
4. A. A. Gol'dberg and I. V. Ostrovskii, The Distributions of the Values of Meromorphic Functions [in Russian], Nauka, Moscow (1970).
5. R. S. Yulmukhametov, "Approximation of subharmonic functions," Mat. Sb., <u>124</u> (166), No. 3, 393-415 (1984).

*In the case a = 0 this theorem has been proved in the diploma work of L. M. Pshenitskii, S. S. Sichak, G. V. Yakimets, carried out in 1985 at the Lvov State University, under the guidance of the first of the authors.