GROWTH OF THE NEVANLINNA PROXIMITY FUNCTION

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Let the function f be meromorphic in the finite plane. Put

$$m(r, f) = m(r, \infty, f) = \frac{1}{2\pi}\int_0^{2\pi} \log^+ |f(re^{i\varphi})|\, d\varphi,$$

$$m(r, a, f) = m(r, 1/(f-a)),\ a \neq \infty.$$

Drasin and Weitsman [1] showed that the set $A \subset \bar{\mathbf{C}}$ of values a for which

$$m(r, a, f) \to \infty, \quad r \to \infty, \tag{1}$$

holds has zero capacity. They also constructed an entire function of order ρ for which (1) holds for $a \in A$ for any set A of zero capacity and any $\rho > 1/2$. The analogous problem was posed by Drasin and Weitsman [2, p. 156, Problem 1.27(c)] for meromorphic functions of order $\rho \leq 1/2$. (It is well known that for entire functions of order $\rho < 1/2$ Eq. (1) is satisfied only for $a = \infty$.)

The following theorem answers the question of Drasin and Weitsman.

THEOREM. Let $A \subset \bar{\mathbf{C}}$ be a set of zero capacity, $\psi(r)$ defined on $[0, \infty)$ and monotonically increasing with $\psi(r) \to \infty$ as $r \to \infty$, $\psi(0) = 1$. There exists a meromorphic function F for which (1) holds with $a \in A$ and

$$T(r, F) = O(\psi(r) \log^2 r),\ r \to \infty \tag{2}$$

(T(r, F) is the Nevanlinna characteristic function of F).

LEMMA. Let $\{\theta_k\}_{k=1}^N$ and $\{\theta'_k\}_{k=1}^N$ be finite sequences of numbers in the interval $(-\pi/2, \pi/2)$; $\theta_k < \theta'_k < \theta_{k+1} < \theta'_{k+1}$. Put $D = \bigcup_{k=1}^N [\theta_k, \theta'_k]$. Let $\{a_k\}_{k=1}^N$ be complex numbers with $|a_k| \leq \sqrt{2}/2$. There exists a meromorphic function f with the properties:

$$T(r, f) = o(\psi(r) \log^2 r), \quad r \to \infty, \tag{3}$$

$$f(z) \to a_k \quad \text{uniformly as} \quad |z| \to \infty,\ \theta_k \leq \arg z \leq \theta'_k, \tag{4}$$

$$f(0) = 0, \tag{5}$$

$$|f(z)| \leq 1, \quad \arg z \in D. \tag{6}$$

Proof. Valiron [3] constructed a meromorphic function g satisfying condition (3) and having the following properties: for every α, $0 < \alpha < \pi/2$, $g(z) \to 1$ uniformly as $|z| \to \infty$, $\alpha \leq \arg z \leq \pi - \alpha$, $g(z) \to 0$ uniformly as $|z| \to \infty$, $\pi + \alpha \leq \arg z \leq 2\pi - \alpha$.

We put $\varphi_1 = (1/2)(\theta_1 - \pi/2)$, $\varphi_k = (1/2)(\theta_k + \theta'_{k-1})$ for $2 \leq k \leq N$. Then the function

$$g_1(z) = a_1 g(ze^{-i\varphi_1}) + \sum_{k=2}^N (a_k - a_{k-1})\, g(ze^{-i\varphi_k})$$

has properties (3), (4). Let $\{b_k\}_{k=1}^K$ be all the poles of the function g_1 on the set $E = \{z : \arg z \in D\}$. (Their number is finite by (4).) The function $g_2(z) = g_1(z)(z+1)^{-K} \prod_{k=1}^K (z - b_k)$ is holomorphic in E. We put $M(z) = \max_{\substack{|z|=r \\ z \in E}} |g_2(z)|$.

It follows from (4) that $\overline{\lim_{r\to\infty}} M(r) \leq \sqrt{2}/2$. Assume that $M(r) \leq 1$ and $r > r_0$. We put $M = \max_{r \leq r_0} M(r)$ and take $\varkappa > 0$ so small that $|\varkappa z(\varkappa z + 1)^{-1}| < 1/M$ holds for $|z| \leq r_0$. Bearing in mind that $|\varkappa z(\varkappa z + 1)^{-1}| < 1$ in the right halfplane, we obtain that the function

$$f(z) = \varkappa z(\varkappa z + 1)^{-1} g_2(z)$$

satisfies all the hypotheses of the lemma.

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Proof of the Theorem. We first prove the theorem in the case when the set A is bounded. Without loss of generality, it is then possible to assume that $A\subset\{z: |z|\leqslant 1/2\}$. By a theorem of Cartan [4, p. 96] there exists a measure μ such that

$$\int\limits_C \log\frac{1}{|\xi-a|}\,d\mu(\xi)=\infty,\quad a\in A.$$

We may assume that the support of μ is contained in the square $\Delta_1^0=\{z=x+iy: -1/2\leq x<1/2, -1/2\leq y<1/2\}$, and that the total measure is equal to $\pi/2$. We divide the square Δ_1^0 into congruent squares Δ_j^1, $j=1,\dots,4$. (The squares are numbered in a clockwise direction starting with the upper left square.)

Assume that the square Δ_1^0 has already been subdivided into congruent squares Δ_j^{n-1}, $j=1,\dots,4^{n-1}$. We divide each of the squares Δ_j^{n-1} into four congruent squares ordered clockwise from the upper left. We arrange the quarter-squares thus obtained in the same order as Δ_j^{n-1} and number them in sequence. We get a sequence Δ_j^n, $j=1,\dots,4^n$. (We assume that the left and lower sides belong to the square Δ_j^n; the right and upper sides do not belong to it.) We note that the length of a side of the square Δ_j^n equals 2^{-n}.

We denote by mes α the radian measure of the angle α. We divide the first quadrant into four angles β_j^1 so that

$$\operatorname{mes}\beta_j^1=\mu(\Delta_j^1).$$

This can be done since $\sum_{j=1}^{4}\mu(\Delta_j^1)=\mu(\Delta_1^0)=\pi/2$. The angles are enumerated counterclockwise. Let α_j^1 be the angle with the same bisectrix as β_j^1 but with

$$\operatorname{mes}\alpha_j^1=\tfrac12\,\mu(\Delta_j^1)=\tfrac12\operatorname{mes}\beta_j^1.$$

We divide each of the angles α_j^{n-1} into four angles β_j^n (numbered in sequence in a counterclockwise direction) with

$$\operatorname{mes}\beta_j^n=\frac{2}{\pi}\operatorname{mes}\left(\bigcup_{k=1}^{4^{n-1}}\alpha_k^{n-1}\right)\mu(\Delta_j^n).$$

Let α_j^n be the angle with the same bisectrix as β_j^n but with

$$\operatorname{mes}\alpha_j^n=(1-2^{-n})\operatorname{mes}\beta_j^n.$$

Putting $T=\bigcap_{n=1}^{\infty}\bigcup_{j=1}^{4^n}\alpha_j^n$, it is easy to see that

$$\operatorname{mes}T=\frac{\pi}{2}\prod_{k=1}^{\infty}(1-2^{-k})=t>0$$

and

$$\operatorname{mes}(\alpha_j^n\cap T)=t\mu(\Delta_j^n). \tag{7}$$

We put $a_1=\frac12(-1+i)$, $a_2=\frac12(1+i)$, $a_3=\frac12(1-i)$, $a_4=\frac12(-1-i)$.

If we apply the lemma to the set $D=\bigcup_{j=1}^{4^n}\alpha_j^n\cup[-\pi/4,0]$, we get a meromorphic function f_n with properties (3), (5), (6), and such that

$$f_n(z)\to a_k,\ \arg z\in\alpha_j^n,\quad j\equiv k \pmod 4, \tag{8}$$

$$f_n(z)\to 0,\quad -\pi/4\leqslant\arg z\leqslant 0, \tag{9}$$

uniformly as $|z|\to\infty$. It follows from (3), (5) that there exist numbers $\varkappa_n>0$ such that

$$|f_n(\varkappa_n z)|\leqslant 1\quad\text{for}\quad |z|\leqslant e^n, \tag{10}$$

$$T(r, f_n(\varkappa_n z))\leqslant 2^{-n}\psi(r)\log^2 r,\quad r\geqslant 2. \tag{11}$$

By (10) the series $F(z)=\sum_{n=1}^{\infty}2^{-n}f_n(\varkappa_n z)$ represents a function meromorphic in the finite plane, and by (10) and (11) the function F satisfies condition (2).

Let ξ_j^m be the midpoint of the square Δ_j^m. It follows from (8) that

$$\sum_{n=1}^{m}2^{-n}f_n(\varkappa_n z)\to\xi_j^m\quad\text{as}\quad |z|\to\infty,\ \arg z\in\alpha_j^m. \tag{12}$$

We show that the function F satisfies the hypothesis of the theorem. From the definition of the measure μ and Fatou's theorem, we obtain

$$S_m(a) = \sum_{j=1}^{4^m} \left(\inf_{\xi \in \Delta_j^m} \log^+ \frac{1}{|\xi - a|} \right) \mu(\Delta_j^m) \to \infty$$

as $m \to \infty$, $a \in A$. Let m be any natural number. By (6) we have

$$\left| \sum_{n=m+1}^{\infty} 2^{-n} f_n(\varkappa_n z) \right| \leqslant 2^{-m} \quad \text{for} \quad \arg z \in T. \tag{13}$$

In accordance with (12) we choose r_0 so that

$$\sum_{n=1}^{m} 2^{-n} f_n(\varkappa_n z) \in \Delta_j^m$$

for $\arg z \in \alpha_j^m$, $|z| \geqslant r_0$. For such z we have by (13) that $F(z) \in D_j^m$, where D_j^m is a square with the same center as Δ_j^m and with sides parallel to the sides of Δ_j^m but three times longer.

It is easy to see that for any a

$$\inf_{\xi \in D_j^m} \log^+ \frac{1}{|\xi - a|} \geqslant \inf_{\xi \in \Delta_j^m} \log^+ \frac{1}{|\xi - a|} - \log 3.$$

Thus for $r > r_0$ we have, bearing in mind (7),

$$m(r, a, F) \geqslant \frac{1}{2\pi} \int_0^{\pi/2} \log^+ \frac{1}{|F(re^{i\varphi}) - a|} d\varphi \geqslant \frac{1}{2\pi} \int_T \log^+ \frac{1}{|F(re^{i\varphi}) - a|} d\varphi \geqslant$$

$$\geqslant \frac{1}{2\pi} \sum_{j=1}^{4^m} \left(\inf_{\xi \in D_j^m} \log^+ \frac{1}{|\xi - a|} \right) \operatorname{mes}(\alpha_j^m \cap T) \geqslant \frac{t}{2\pi} \left(S_m(a) - \frac{\pi \log 3}{2} \right). \tag{14}$$

Since m was chosen arbitrarily and $S_m(a) \to \infty$ as $m \to \infty$, it follows that $m(r, a, F) \to \infty$ for $a \in A$.

We now consider the case when the set A is unbounded. Let $A = A_1 \cup A_2, A_1 \subset \{z: |z| \leqslant 1\}$, $A_2 \subset \{z: |z| > 1\}$. We put $A' = \{a: 1/a \in A_2\}$. Since the theorem has been proved for any bounded set A, there exists a meromorphic function F_1 satisfying condition (2) and by (9), (14) having the properties:

$$\int_0^{\pi/2} \log^+ \frac{1}{|F_1(re^{i\varphi}) - a|} d\varphi \to \infty \quad \text{as} \quad r \to \infty, \quad \text{if} \quad a \in A_1, \tag{15}$$

$$F_1(z) \to \text{uninformly as} \qquad |z| \to \infty, \ -\pi/4 \leqslant \arg z \leqslant 0. \tag{16}$$

Analogously, we construct a meromorphic function F_2 satisfying (2) with the properties

$$\int_{-\pi/4}^{-\pi/8} \log^+ \frac{1}{|F_2(re^{i\varphi}) - a|} d\varphi \to \infty \quad \text{as} \quad r \to \infty, \quad \text{if} \quad a \in A', \tag{17}$$

$$F_2(z) \to 0 \quad \text{uniformly as} \quad |z| \to \infty, \ 0 \leqslant \arg z \leqslant \pi/2. \tag{18}$$

We now consider a function f satisfying (2) such that

$$f(z) \to 0, \qquad -\pi/4 \leqslant \arg z \leqslant -\pi/8, \tag{19}$$

$$f(z) \to \infty, \qquad 0 \leqslant \arg z \leqslant \pi/2, \tag{20}$$

uniformly as $|z| \to \infty$.

We show that the function

$$F(z) = F_1(z) + 1/[F_2(z) + f(z)],$$

for which (2) obviously holds, satisfies the hypothesis of the theorem. Let g be a meromorphic function tending to zero as $|z| \to \infty$, $0 \leq \arg z \leq \pi/2$. It is easy to see using (14) that Eq. (15) will hold if F_1 is replaced by $F_1 + g$. If, on the other hand, the function g tends to zero uniformly as $|z| \to \infty$, $-\pi/4 \leq \arg z \leq -\pi/8$, relation (17) will hold for the function $F_2 + g$.

We let $k(a, b)$ denote the chordal distance between the points a and b on the Riemann sphere. It is easy to see that

$$k(a,\ b)=k(1/a,\ 1/b). \tag{21}$$

Moreover, for any θ_1 and θ_2 we have as $r \to \infty$

$$\int_{\theta_1}^{\theta_2} \log \frac{1}{k(F(re^{i\varphi}),\ a)}\, d\varphi = \int_{\theta_1}^{\theta_2} \log^{+} \frac{1}{|F(re^{i\varphi}) - a|}\, d\varphi + O(1). \tag{22}$$

If $a \in A_1$, then by (18), (20), (15) we obtain in succession

$$\frac{1}{2\pi}\int_{0}^{\pi/2} \log^{+} \frac{1}{|F_1(re^{i\varphi}) - a|}\, d\varphi \to \infty,\ r \to \infty.$$

$$m(r,\ a,\ F) \geqslant \frac{1}{2\pi}\int_{0}^{\pi/2} \log^{+} \frac{1}{|F(re^{i\varphi}) - a|}\, d\varphi \to \infty,\ r \to \infty.$$

If, on the other hand, $a \in A_2$, we obtain in succession (17), (22), (21), (19), (16). We have

$$\frac{1}{2\pi}\int_{-\pi/4}^{-\pi/8} \log^{+} \frac{1}{|F_2(re^{i\varphi}) - a^{-1}|}\, d\varphi \to \infty,$$

$$\frac{1}{2\pi}\int_{-\pi/4}^{-\pi/8} \log \frac{1}{k(F_2(re^{i\varphi}),\ a^{-1})}\, d\varphi \to \infty,$$

$$\frac{1}{2\pi}\int_{-\pi/4}^{-\pi/8} \log \frac{1}{k((F_2(re^{i\varphi}) + f(re^{i\varphi}))^{-1},\ a)}\, d\varphi \to \infty,$$

$$m(r,\ a,\ F) \geqslant \frac{1}{2\pi}\int_{-\pi/4}^{-\pi/8} \log^{+} \frac{1}{|F(re^{i\varphi}) - a|}\, d\varphi \to \infty,\ r \to \infty.$$

The theorem is proved.

Remark 1. Let the meromorphic function f satisfy the condition $\varliminf_{r\to\infty} T(r, f)/\ln^2 r < \infty$. In this case, as was shown by Tumura [5], there exists a sequence of positive numbers $\tau_k \to \infty$, such that the sequence $f(\tau_k z)$ converges uniformly in the annulus $\{z : 1 \le |z| \le 2\}$. It follows that Eq. (1) can hold for the function f for at most one value $a \in \bar{C}$.

Remark 2. Assume that a meromorphic function of a given order ρ, $0 \le \rho \le \infty$, tends to zero uniformly in the right halfplane. If we add this function to the function F constructed in this paper, we obtain a meromorphic function of order ρ for which (1) holds.

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Remark. While this paper was in press, the following somewhat weaker result of Damodaran [6] appeared: for every set $A \subset \bar{C}$ of zero capacity and any function $\varphi(r)$ tending to infinity, there exists a meromorphic function f with property (1) for all $a \in A$ and $T(r, f) = O(\varphi(r) \ln^3 r)$, $r \to \infty$.

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