SET OF ASYMPTOTIC VALUES OF A MEROMORPHIC FUNCTION OF FINITE ORDER

A. É. Eremenko

It is well known that an entire function of finite order has an at most countable number of asymptotic values. Gross [1] constructed an example of an entire function of infinite order whose set of asymptotic values coincides with the extended complex plane. Dreisin and Weizmann [2] raised the question of whether there are any restrictions from above on the size of the set of asymptotic values of a meromorphic function of finite order. In this direction only the result of Valiron [3] is known; he constructed a meromorphic function of finite order with a set of asymptotic values having the cardinality of the continuum. In this paper we construct a meromorphic function of finite order whose set of asymptotic values coincides with the extended complex plane, thereby giving a negative answer to the question of Dreisin and Weizmann.

We express the numbers in [0, 1] in base seven and reject expressions ending in an infinite sequence of sixes. Let α_n, β_n, γ_n, μ_n denote the sets of numbers in [0, 1] whose expressions contain 0, 2, 4, 6, respectively, in the n-th place. We write $\bar{A}$ for the closure of a set A on the real axis and put

$$E_n = \bar{\alpha}_n \cup \bar{\beta}_n \cup \bar{\gamma}_n \cup \bar{\mu}_n,$$

$$E_n' = E_n \cup [\pi - 1, \pi].$$

<u>LEMMA.</u> For each positive integer n there exists a meromorphic function f_n of the first order with the properties

$$|f_n(z)| \leqslant 2, \quad \arg z \in E_n', \tag{1}$$

$$f_n(0) = 0, \tag{2}$$

$$\left.\begin{array}{ll} f_n(z) \to 1, & \arg z \in \bar{\alpha}_n, \\ f_n(z) \to i, & \arg z \in \bar{\beta}_n, \\ f_n(z) \to 1+i, & \arg z \in \bar{\gamma}_n, \\ f_n(z) \to 0, & \arg z \in \bar{\mu}_n \cup [\pi-1, \pi] \end{array}\right\} \tag{3}$$

uniformly in arg z as $|z| \to \infty$.

<u>Proof.</u> The set E_n' consists of finitely many segments. Let $\{\theta_j\}_{j=1}^N$ be the midpoints of these segments. It is easy to see that if θ_j, θ_{j+1} are the midpoints of two adjacent segments in E_n' then $(1/2)(\theta_j + \theta_{j+1}) \bar{\in} E_n'$. Consider the function

$$g(z) = \frac{\sum_{j=1}^N a_j \exp(ze^{-i\theta_j})}{\sum_{j=1}^N \exp(ze^{-i\theta_j})},$$

where $a_j = 1$ for $\theta_j \in \alpha_n$; $a_j = i$ for $\theta_j \in \beta_n$; $a_j = 0$ for $\theta_j \in \mu_n \cup [\pi - 1, \pi]$; $a_j = 1 + i$ for $\theta_j \in \gamma_n$. It is known (see [4, p. 161]) that the function g, which is clearly of the first order, possesses property (3). Let $\{b_k\}_{k=1}^m$ be all the poles of g on the set $D = \{z\colon \arg z \in E_n'\}$ (there are finitely many of them by (3)). The function

$$h(z) = g(z)(z+i)^{-m} \prod_{k=1}^m (z - b_k) \tag{4}$$

has no poles in D and has property (3) along with g. We put

$$M(r) = \max\{|h(z)|\colon z \in D, |z| = r\}.$$

Lvov State University. Translated from Matematicheskie Zametki, Vol. 24, No. 6, pp. 779-783, December, 1978. Original article submitted October 6, 1976.

 0001-4346/78/2456-0914$07.50

By (3) we have $\overline{\lim}_{r\to\infty} M(r) = \sqrt{2}$. Let r_0 be such that $M(r) < 2$ for $r > r_0$. We put $M = \max\{M(r): r \leqslant r_0\}$ and let $\eta > 0$ be so small that for $|z| \leqslant r_0$ we have $|\eta z(\eta z + i)^{-1}| < 2/M$. Using the fact that $|\eta z(\eta z + i)^{-1}| < 1$ in the upper halfplane, we obtain that the function $f_n(z) = \eta z(\eta z + i)^{-1} h(z)$ satisfies all the conditions of the lemma.

THEOREM. There exists a meromorphic function of finite order for which every value $a \in \overline{C}$ is asymptotic.

Proof. Consider the sequence of functions f_n constructed in the lemma. Let $T(r, f)$ be the Nevanlinna characteristic of f. Then it is easy to see that there exist numbers $r_n > 0$ such that

$$T(r, f_n) < 2^{-n} r^2, \quad r > r_n. \tag{5}$$

In addition, we can find by (2) a δ_n, $0 < \delta_n < 1$, sufficiently small, such that

$$|f_n(\delta_n z)| < 1 \text{ for } |z| < \max(n, r_n). \tag{6}$$

It follows from (4) and (5) that for all $r > 0$

$$T(r, f_n(\delta_n z)) < 2^{-n} r^2. \tag{7}$$

The series $F(z) = \sum_{n=1}^{\infty} 2^{-n} f_n(\delta_n z)$ converges uniformly on compact sets by (6) and therefore represents a meromorphic function in the finite plane. It follows from (6), (7) that

$$T(r, F) \leqslant T\left(r, \sum_{n=1}^{[r]} 2^{-n} f_n(\delta_n z)\right) + T\left(r, \sum_{n=[r]+1}^{\infty} 2^{-n} f_n(\delta_n z)\right) \leqslant$$
$$\leqslant \sum_{n=1}^{[r]} T(r, f_n(\delta_n z)) + O(\log r) \leqslant 2r^2 + O(\log r), \quad r \to \infty.$$

Consequently, the order of F is at most 2.

We show that every complex number in the square

$$\Delta = \{z: 0 \leqslant \operatorname{Re} z \leqslant 1, \ 0 \leqslant \operatorname{Im} z \leqslant 1\}$$

is an asymptotic value for F. Every $a \in \Delta$, can be written in the form $a = \sum_{n=1}^{\infty} 2^{-n} \nu_n$, where the ν_n take one of the values 0, 1, 1 + i, i. We define a number $\varphi_a \in [0, 1]$ using the base-seven expansion

$$\varphi_a = 0, t_1, t_2, \ldots,$$

where $t_j = 0$ for $\nu_j = 1$; $t_j = 2$ for $\nu_j = i$; $t_j = 4$ for $\nu_j = 1 + i$; $t_j = 6$ for $\nu_j = 0$. We show that $F(z) \to a$ for $\arg z = \varphi_a$, $|z| \to \infty$. Let $\varepsilon > 0$ be arbitrary and choose a number N such that

$$\left|\sum_{n=N+1}^{\infty} 2^{-n} \nu_n\right| < \varepsilon/3, \tag{8}$$

$$\left|\sum_{n=N+1}^{\infty} 2^{-n} f_n(\delta_n z)\right| < \varepsilon/3. \tag{9}$$

By (1) we can arrange it so that (9) holds, since $\varphi_a \in \bigcap_{n=1}^{\infty} E_n'$. By (3), we can now choose r_0 so large that when $\arg z = \varphi_a$, $r > r_0$

$$\sum_{n=1}^{N} |f_n(\delta_n z) - \nu_n| < \varepsilon/3. \tag{10}$$

Combining (8), (9), and (10) we obtain for $\arg z = \varphi_a$, $|z| > r_0$: that

$$|F(z) - a| = \left|\sum_{n=1}^{\infty} 2^{-n} f_n(\delta_n z) - \sum_{n=1}^{\infty} 2^{-n} \nu_n\right| < \varepsilon.$$

Hence $F(z) \to a$ for $\arg z = \varphi_a$, $|z| \to \infty$. It is proved in exactly the same way that $F(z) \to 0$ as $|z| \to \infty$, uniformly for $\pi - 1 \leqslant \arg z \leqslant \pi$.

Putting $G_1 = 2F - 1 - i$, it follows from what has been proved that the asymptotic values of the function G_1 cover the closed unit disk centered at the coordinate origin, and the corresponding asymptotic paths lie in the sector $\{z: 0 \leqslant \arg z \leqslant 1\}$. It is possible to use the same method to construct a meromorphic function G_2 whose asymptotic values cover the closed unit disk, the corresponding asymptotic paths lying in the sector $\{z: \pi - 1 \leqslant \arg z \leqslant \pi\}$, and such that in addition $G_2(z) \to 0$ as $|z| \to \infty$, uniformly for $0 \leqslant \arg z \leqslant 1$. Thus the asymptotic values of the meromorphic function $(G_2(z) + e^z)^{-1}$, corresponding to the asymptotic paths lying inside the sector $\{z: \pi - 1 \leqslant \arg z \leqslant \pi\}$, cover the exterior of the closed unit disk. Hence the function

$$G(z) = G_1(z) + (G_2(z) + e^z)^{-1},$$

which has order at most 2, satisfies the conditions of the theorem.

Remark. One can construct a meromorphic function satisfying the condition of the theorem and having a given order $\rho > 0$. In order to do this we make use of the well known approach (see, e.g., [4, p. 164]) and construct as above a meromorphic function of order at most ρ_1, $0 < \rho_1 < \min(1, \rho)$, to which we then add an arbitrary meromorphic function of order ρ which tends to zero as $|z| \to \infty$ uniformly in $0 \leqslant \arg z \leqslant \pi$. It is also possible to find the smallest possible growth of the characteristic of a function satisfying the condition of the theorem.

Let $\psi(r)$ be an arbitrary positive function tending monotonely to infinity as $r \to \infty$. In [3], Valiron constructed a meromorphic function g^* having properties (3) such that

$$T(r, g^*) = o(\psi(r) \log^2 r), \quad r \to \infty$$

is satisfied. Replacing the function (4) in our construction by g^*, we obtain a meromorphic function satisfying the condition of the theorem and in addition the condition

$$T(r, G) = O(\psi(r) \log^2 r), \quad r \to \infty.$$

On the other hand, Valiron [5, Sec. 33] proved that if the condition $T(r, G) = O(\log^2 r)$, holds as $r \to \infty$, then the function G has at most one asymptotic value.

The author thanks A. A. Gol'dberg for his guidance of this work and for a number of valuable remarks.

LITERATURE CITED

1. W. Gross, "Eine ganze Funktion für die jede komplexe Zahl Konvergenzwert ist," Math. Ann., 79, 201-208 (1918).
2. K. Hayman, "New problems," in: Symposium on Complex Analysis, Canterbury, 1973, Cambridge Univ. Press (1974), pp. 155-180.
3. G. Valiron, "Sur les valeurs asymptotiques de quelques fonctions méromorphes," Rend. Circ. Mat. Palermo, 49, No. 3, 415-421 (1925).
4. A. A. Gol'dberg and I. V. Ostrovskii, Value Distribution of Meromorphic Functions [in Russian], Nauka, Moscow (1970).
5. G. Valiron, "Fonctions entières d'ordre fini et fonctions méromorphes. Fonctions entières d'ordre fini et d'ordre null. VI, V," Enseign. Math., 4, No. 3, 157-177 (1958).