Consequently,

$$\lim_{n\to\infty} b_{n,k} = F^*(I-P)(TP)^k F = b_k,\ k=0,\ 1,\ 2,\ \ldots,$$

and the lemma is entirely proved.

The role of the obtained factorizations at the investigations of contractive matrix functions will be elucidated in the next part of the paper.

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MINIMUM OF THE MODULUS OF AN ENTIRE FUNCTION ON THE SEQUENCE OF POLYA PEAKS

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Let f be an entire function of genus zero, let (r_k) be a sequence of Pólya peaks for N(r, f) of order $\lambda < 1$. Then there exists a sequence $r'_k \sim r_k$ such that
$$\min_{|z|=r'_k} \ln|f(z)| \geq (\cos\pi\lambda + o(1)) \ln M(r'_k, f),\ k\to\infty.$$
If for (r_k) one takes a sequence of Pólya peaks for $\ln M(r, f)$ or for $T(r, f)$, $^1/_2 < \lambda < 1$, then the result ceases to be true.

For a transcendental entire function f we set

$$L(r, f) = \inf\{|f(z)| : |z| = r\},\ M(r, f) = \sup\{|f(z)| : |z| = r\}.$$

B. Kjellberg has shown that the classical Wiman–Valiron inequality

$$\limsup_{r\to\infty} \frac{\ln L(r, f)}{\ln M(r, f)} \geq \cos\pi\lambda \tag{0.1}$$

is satisfied if the lower order λ of the function f does not exceed one (see, for example, [1, Chap. V, Theroem 3.4]. A large number of investigations have been devoted to various refinements and generalization of this inequality. In this paper we prove a certain refinement of the estimate (0.1) and we also refute several conjectures connected with this estimate.

1. Fundamental Results. We give some definitions.

A sequence $r_k \to \infty$ is said to be a sequence of Pólya peaks [of the first kind] of order μ for an unboundedly increasing positive function S(r) if there exists a sequence $\eta_k \to 0$ such that

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$$S(r) \leqslant (1+\eta_k) S(r_k) \left(\frac{r}{r_k}\right)^{\mu}; \quad \eta_k r_k \leqslant r \leqslant \eta_k^{-1} r_k. \tag{1.1}$$

It is known [2] that such a sequence exists if $\lambda_* \le \mu \le \rho^*$, where

$$\lambda_* = \lambda_*(S) = \inf\left\{\mu : \liminf_{r,\, t\to\infty} \frac{S(tr)}{S(r)\, t^{\mu}} < \infty\right\};$$

$$\rho^* = \rho^*(S) = \sup\left\{\mu : \limsup_{r,\, t\to\infty} \frac{S(tr)}{S(r)\, t^{\mu}} > 0\right\}.$$

If T(r, f) is the Nevanlinna characteristic of the entire function f, then $\lambda_*(T) = \lambda_*(\ln M)$, $\rho^*(T) = \rho^*(\ln M)$; in the sequel these numbers will be denoted by λ_* and ρ^*.

The sequence r_k is said to be a sequence of strong Polya peaks of order μ for an entire function f if (1.1) is satisfied with S(r) = N(r, f), where N(r, f) is a the Nevanlinna function of the number of zeros of the entire function f, and, moreover

$$\ln M(r, f) \leqslant CN(r_k, f)\left(\frac{r}{r_k}\right)^{\mu}; \quad \eta_k r_k \leqslant r \leqslant \eta_k^{-1} r_k. \tag{1.2}$$

(Here and in the sequel, by C we denote various positive constants). Such a sequence always exists [3] if μ is not an integer and $\lambda_* \le \mu \le \rho^*$.

<u>THEOREM 1.</u> Let (r_k) be an arbitrary sequence of strong Polya peaks of order $\mu > 1$ for the entire function f. Then there exists a sequence $r'_k \sim r_k$ such that

$$\liminf_{k\to\infty} \frac{\ln L(r'_k, f)}{\ln M(r'_k, f)} \geqslant \cos \pi\mu. \tag{1.3}$$

We give two proofs of Theorem 1. The first one is based on the passage from the function f to the limiting subharmonic function in the sense of V. S. Azarin [4] and on the proof for it of the "nonasymptotic form" of Theorem 1 [Theorem 1a]. In a related situation this method has been applied by J. M. Anderson and A. Baernstein. The second way of proving Theorem 1 is based on standard methods.

In conversations with one of the authors, A. Edrei has posed the question whether the estimate (1.3) holds if (r_k) is a sequence of Pólya peaks of order μ for T(r, f). In this case, A. Edrei has proved* that for each $\alpha > \mu$ there exists a constant $C = C(\alpha, \mu)$ and a sequence (r'_k), $r_k \le r'_k \le Cr_k$ such that

$$\limsup_{k\to\infty} \frac{\ln L(r'_k, f)}{\ln M(r'_k, f)} \geqslant \cos \pi\alpha.$$

For $\mu \le 1/2$ the answer to A. Edrei's question is in the affirmative by virtue of Theorem 1. Indeed, in this case one can show that the strong Polya peaks coincide with the Pólya peaks for T(r, f). For $1/2 < \mu < 1$ the answer to A. Edrei's question is negative.

<u>Example 1.</u> For each μ, $1/2 < \mu < 1$, there exists an entire function f for which $\lambda_* = \rho^* = \mu$, the sequences of Pólya peaks for T(r, f) and for ln M(r, f) coincide and, moreover, for any such sequence (r_k) we have

*A. Edrei, A local form of the Phragmén–Lindelöf indicator. Mathematika, Vol. 17, pp. 149-172, 1970.

$$\limsup_{k\to\infty} \sup\left\{\frac{\ln L(r, f)}{\ln M(r, f)} : \sigma^{-1}r_k \leq r \leq \sigma r_k\right\} \leq -q(\sigma), \tag{1.4}$$

where $q(\sigma) \to +\infty$ for $\sigma \to 1$.

D. Drasin and A. Weitsman [5, No. 2.37] have suggested to elucidate whether relation (0.1) holds in the neighborhoods of the Polya peaks of order λ for the function $\ln M(r, f)$. A negative answer to this question is given by

Example 2. For each μ, $0 < \mu < 1$, there exists an entire function f for which $\lambda_* = \rho^* = \mu$ and, moreover, on any sequence of Polya peaks (r_k) for $\ln M(r, f)$ inequality (1.4) is satisfied.

2. A Nonasymptotic Form of Theorem 1. For a subharmonic function u we set

$$A(r, u) = \inf\{u(z) : |z| = r\}; \quad B(r, u) = \max\{u(z) : |z| = r\};$$
$$N(r, u) = \frac{1}{2\pi}\int_0^{2\pi} u(re^{i\theta})\, d\theta,$$

and by ν we denote the Riesz measure of the function u, $\nu(t) = \nu(\{z : |z| \leq t\})$. If $u(0) = 0$, then $N(r, u) = \int_0^r \frac{\nu(t)}{t}\, dt$.

THEOREM 1a. Let u be a subharmonic function in $\mathbf{C}$, of order less than one. If $u(0) = 0$ and

$$\max\left\{\frac{N(r, u)}{r^\mu},\ 0 < r < \infty\right\} = N(1, u) = 1 \tag{2.1}$$

for some $\mu < 1$, then

$$\frac{A(1, u)}{B(1, u)} \geq \cos \pi\mu. \tag{2.2}$$

Proof. Without loss of generality, we assume that $B(1, u) = u(1)$. The function u admits the representation

$$u(z) = \int_{\mathbf{C}} \ln\left|1 - \frac{z}{\xi}\right| d\nu(\xi). \tag{2.3}$$

If we set

$$v(z) = \int_0^\infty \ln\left|1 + \frac{z}{t}\right| d\nu(t), \tag{2.4}$$

then

$$v(-r) \leq A(r, u) \leq B(r, u) \leq v(r). \tag{2.5}$$

We consider the function

$$v^*(re^{i\theta}) = \frac{1}{2\pi}\int_{-\theta}^{\theta} v(re^{i\varphi})\,d\varphi,\ 0 \leqslant \theta \leqslant \pi,$$

harmonic in the upper halfplane and continuous in its closure. We have $v^*(r) = 0$, $v^*(-r) = N \times (r, u) \leqslant r^\mu$, $0 < r < \infty$. By the Phragmén–Lindelöf theorem we have

$$v^*(re^{i\theta}) \leqslant r^\mu \frac{\sin\mu\theta}{\sin\pi\mu} \equiv r^\mu H(\theta),\ 0 \leqslant \theta \leqslant \pi,\ 0 \leqslant r < \infty. \tag{2.6}$$

We denote $v_\theta^*(re^{i\theta}) = \frac{\partial}{\partial\theta} v(re^{i\theta})$. We have v*(1) = H(0) = 0,*(−1) = H(π) = 1. Therefore, by virtue of (2.5), (2.6) have

$$B(1, u) \leqslant v(1) = \pi v_\theta^*(1) \leqslant \pi H'(+0) = \frac{\pi\mu}{\sin\pi\mu}; \tag{2.7}$$

$$A(1, u) \geqslant v(-1) = \pi v_\theta^*(-1) \geqslant \pi H'(\pi - 0) = \pi\mu \operatorname{ctg}\pi\mu, \tag{2.8}$$

from where we obtain (2.2) if $\mu \leq 1/2$.

If $1/2 < \mu < 1$, then we set $2w(z) = v(i\sqrt{z}) + v(-i\sqrt{z})$. Further, let $\dot{A}(1, u) = u(e^{i\varphi})$. Then, by virtue of (2.3), (2.4) we have

$$A(1, u) + B(1, u) \geqslant u(e^{i\varphi}) + u(e^{-i\varphi}) = \int_C \ln\left|1 - \frac{e^{2i\varphi}}{\zeta^2}\right| d\nu(\zeta) \geqslant \int_0^\infty \ln\left|1 - \frac{1}{t^2}\right| d\nu(t)$$

$$= v(1) + v(-1) = 2w(-1); \tag{2.9}$$

$$\frac{N(r, w)}{r^{\mu/2}} = \frac{N(r^{1/2}, u)}{r^{\mu/2}} \leqslant N(1, w) = 1. \tag{2.10}$$

By virtue of (2.10), the inequalities (2.7), (2.8) are satisfied with the replacement of u by w and of μ by μ/2. Now from (2.9), (2.8), (2.7) there follows

$$A(1, u) + B(1, u) \geqslant 2w(-1) \geqslant \pi\mu \operatorname{ctg}\frac{\pi\mu}{2} =$$

$$= 2\cos^2\frac{\pi\mu}{2}\,\frac{\pi\mu}{\sin\pi\mu} \geqslant 2\cos^2\frac{\pi\mu}{2} B(1, u) = (1 + \cos\pi\mu)\,B(1, u),$$

and Theorem 1a is proved.

In order to derive Theorem 1 from Theorem 1a we make use of some properties of the linear measure.

<u>3. Linear Carleson Measure and Its Properties.</u> Let E be a bounded set in $\mathbb{C}$. We consider coverings of the set E by countable collections of circles of radii r_ν and we set $\ell(E) = \inf\sum_\nu r_\nu$ where the infimum is taken over all such coverings. The linear measure ℓ has the following properties: a) monotonicity: $A \subset B \Rightarrow l(A) \leqslant l(B)$; b) σ-semiadditivity: $A \subset \bigcup_{n=1}^\infty A_n \Rightarrow l(A) \leqslant \sum_{n=1}^\infty l(A_n)$.

We say that a sequence of functions u_n converges to the function u with respect to the linear measure $(u_n \Rightarrow u)$, if for each ε > 0 we have $l(|u_n - u| \geqslant \varepsilon) \to 0$, $n \to \infty$.

We have the analogues of the classical theorems of F. Riesz and Egorov.

LEMMA 1. Let $u_n \Rightarrow u$. Then there exists a sequence (n_k) such that a) $u_{n_k} \to u$ outside some set of zero linear measure; b) for each $\delta > 0$ there exists a set Q_δ such that $\ell(Q_\delta) < \delta$ and $u_{n_k} \rightrightarrows u$ uniformly outside Q_δ.

Proof. We repeat the usual arguments. Let $\varepsilon_n \to 0$. We construct a sequence (n_k) inductively, setting $n_k > n_{k-1}$ so that $l(|u_{n_k} - u| \geqslant \varepsilon_k) < 2^{-k}$. We show that this sequence is the required one. Let

$$R_i = \bigcup_{k=i}^{\infty} (|u_{n_k} - u| \geqslant \varepsilon_k); \quad Q = \bigcap_{i=1}^{\infty} R_i.$$

By virtue of the σ-semiadditivity, we have $l(R_i) \leqslant \sum_{k=1}^{\infty} 2^{-k} = 2^{-i+1}$. Since $R_1 \supset R_2 \supset \ldots \supset R_i \supset \ldots$, by virtue of the monotonicity we have $\ell(Q) = 0$. If $z \notin Q$, then there exists an index j, such that $z \notin R_j$, i.e.,

$$\forall\, k \geqslant j \ |u_{n_k}(z) - u(z)| < \varepsilon_k, \tag{3.1}$$

consequently, $u_{n_k}(z) \to u(z)$, $k \to \infty$.

In order to obtain b), we select $i > \log_2 \frac{1}{\delta} + 1$ and we set $Q_\delta = R_i$. By virtue of (3.1), $u_{n_k} \rightrightarrows u$ unformly with respect to $z \notin Q_\delta$, while $l(Q_\delta) = l(R_i) \leqslant 2^{-i+1} = \delta$. The lemma is proved.

4. Proof of Theorem 1. We assume that $\mu > 0$. The case $\mu = 0$ requires simpler but separate arguments. Without loss of generality, we assume that $f(0) = 1$. We consider the sequence of subharmonic functions $u_n(z) = \ln|f(r_n z)|/N(r_n, f)$. By virtue of (1.2) we have

$$B(r, u_n) \leqslant Cr^\mu, \quad \eta_n < r < \frac{1}{\eta_n}, \quad n \to \infty. \tag{4.1}$$

In addition, $u_n(0) = 0$. By V. S. Azarin's theorem [4], the family (u_n) is precompact, i.e., there exist a subharmonic function u and a sequence (n_k) such that

$$u_{n_k} \underset{D'}{\to} u, \quad k \to \infty, \tag{4.2}$$

in the topology of the Schwartz space $D'(\boldsymbol{R}^2)$ of generalized functions.

By another theorem of V. S. Azarin [4, Theorem 4.4.1], from (4.2) there follows that on each compactum $E \subset \boldsymbol{R}^2$ we have $u_n \Rightarrow u$. Making use of Lemma 1, we obtain

$$A(r, u) = \lim_{k\to\infty} A(r, u_{n_k}); \quad B(r, u) = \lim_{k\to\infty} B(r, u_{n_k}), \tag{4.3}$$

for almost all $r \in (0, \infty)$, where one has to thin out again the sequence (n_k).

We show that every limit function u satisfies the conditions of Theorem 1a. We fix $r \in (0, \infty)$. We have

$$N(r, u) = \lim_{k\to\infty} N(r, u_{n_k}) \leqslant r^\mu, \quad N(1, u) = 1.$$

By virtue of (4.1) and (4.3) we have $B(r, u) \leq Cr^\mu$, and by virtue of the "lifting principle" we have

$$u(0) \geqslant \limsup_{k\to\infty} u_{n_k}(0) = 0.$$

Therefore, $u(0) = 0$.

Assume that Theorem 1 is not true. Then by virtue of (4.3) there exists $\sigma > 1$ such that for all $r \in (\sigma^{-1}, \sigma)$ we have $A(r, u)/B(r, u) \le \cos \pi\mu - \varepsilon$, $\varepsilon > 0$. This contradicts Theorem 1a. Theorem 1 is proved.

Remark. One can show that equality in (2.2) implies

$$u(re^{i\theta}) \equiv \frac{\pi\mu}{\sin \pi\mu} \cos \mu(\theta + \alpha) r^{\mu}, \quad -\pi - \alpha \leqslant \theta \leqslant \pi - \alpha.$$

This leads to the known description of the functions for which equality prevails in the $\cos \pi\lambda$ theorem.

5. Second Proof of Theorem 1. We prove a somewhat more refined statement.

THEOREM 1b. Let f be an entire function and let (r_k) be a sequence of strong Pólya peaks of order $\mu < 1$. Then for every sequence $\sigma_k \to 1 + 0$ such that $(\sigma_k - 1)r_k \to \infty$ there exists a set

$$I_k \subset \left[\frac{r_k}{\sigma_k},\ r_k\sigma_k\right],\ |I_k| \sim r_k\left(\sigma_k - \frac{1}{\sigma_k}\right),\ k \to \infty, \tag{5.1}$$

for which we have

$$\liminf_{k\to\infty,\ r\in I_k} \frac{\ln L(r, f)}{\ln M(r, f)} \geqslant \cos \pi\mu. \tag{5.2}$$

Proof. Making use of a known representation [1, Chap. V, Lemma 3.1] and of (1.2), for $|z| \le 2r_k$

$$\ln|f(z)| = \sum_{|z_n| < r_k/(2\eta_k)} \ln\left|1 - \frac{z}{z_n}\right| + o(N(r_k)). \tag{5.3}$$

where η_k are the numbers from the definition of the strong Polya peaks, while (z_n) are the zeros of the entire function f. Here and until the end of the proof the symbols o, O refer to $k \to \infty$.

We select the sequences

$$\tau_k = 1 + \alpha_k \to 1 + 0,\ \sigma_k = 1 + \beta_k \to 1 + 0,\ k \to \infty,$$

so that we should have

$$\eta_k \leqslant \beta_k < \alpha_k < 1; \tag{5.4}$$

$$\alpha_k^{3/2}/\beta_k \to \infty,\ k \to \infty; \tag{5.5}$$

$$\alpha_k^{1/2} \ln \beta_k \to 0,\ k \to \infty. \tag{5.6}$$

Let $n(r) = n(r, f)$ be the number of the zeros of the function f in the circle $\{z: |z| \le r\}$. We show that

$$\mu-\delta_k<\frac{n(tr_k)}{N(r_k)}<\mu+\delta_k,\quad \frac{1}{\tau_k}<t<\tau_k, \tag{5.7}$$

where

$$\delta_k=O(\alpha_k^{1/2}). \tag{5.8}$$

We prove, for example, the right-hand side of the estimates (5.7). For $\tau_k r_k \le r \le r_k/\eta_k$:

$$n(\tau_k r_k)\ln\frac{r}{\tau_k r_k}\leqslant \int_{\tau_k r_k}^{r}\frac{n(t)}{t}\,dt=N(r)-N(\tau_k r_k)\leqslant \eta_k N(r_k)\left(\frac{r}{r_k}\right)^{\mu}+N(r_k)\left\{\left(\frac{r}{r_k}\right)^{\mu}-1\right\}.$$

Setting here $r=r_k(1+\alpha_k^{1/2})$ and making use of (5.6), we obtain

$$\frac{n(\tau_k r_k)}{N(r_k)}\leqslant\frac{\eta_k(1+\alpha_k^{1/2})^{\mu}+(1+\alpha_k^{1/2})^{\mu}-1}{\ln(1+\alpha_k^{1/2})-\ln(1+\alpha_k)}\leqslant\mu+O(\alpha_k^{1/2}).$$

The left-hand side of (5.7) is proved in a similar manner.

We show that for $r\in[r_k/\sigma_k,\ r_k\sigma_k]\setminus E_k=I_k,\ |E_k|=o\left(r_k\left(\sigma_k-\frac{1}{\sigma_k}\right)\right)$ we have

$$\liminf_{r\in I_k,\ k\to\infty}\frac{\ln L(r,f)}{N(r,f)}\geqslant\pi\mu\operatorname{ctg}\pi\mu. \tag{5.9}$$

We estimate ln L(r, f) from below. We set $\sum_k=\sum\ln\left|1-\frac{r}{|z_n|}\right|$, where the summation is carried out over the zeros z_n: $r_k/\tau_k\leqslant|z_n|\leqslant r_k\tau_k$. By virtue of the equality (5.3), integrating by parts, we obtain

$$\begin{aligned}\ln L(r,f)&\geqslant\sum_{|z_n|<r_k/(2\eta_k)}\ln\left|1-\frac{r}{|z_n|}\right|+o(N(r_k))=\\&=\left(\int_0^{r_k/\tau_k}+\int_{r_k\tau_k}^{r_k/(2\eta_k)}\right)\ln\left|1-\frac{r}{t}\right|dn(t)+\sum\nolimits_k+o(N(r_k))=\\&=\left(\int_0^{r_k/\tau_k}+\int_{r_k\tau_k}^{r_k/(2\eta_k)}\right)\frac{r}{r-t}\,dN(t)+n\left(\frac{r_k}{2\eta_k}\right)\ln\left(1-\frac{r}{r_k/(2\eta_k)}\right)+\\&+n\left(\frac{r_k}{\tau_k}\right)\ln\left(\frac{r}{r_k/\tau_k}-1\right)-n(r_k\tau_k)\ln\left(1-\frac{r}{r_k\tau_k}\right)+\sum\nolimits_k+o(N(r_k)).\end{aligned} \tag{5.10}$$

We estimate from below the contribution of the insets. By virtue of the estimates (5.5), (5.7), (5.8), we have

$$\begin{aligned}&n\left(\frac{r_k}{\tau_k}\right)\ln\left(\frac{r}{r_k/\tau_k}-1\right)-n(r_k\tau_k)\ln\left(1-\frac{r}{r_k\tau_k}\right)\geqslant\\&\geqslant N(r_k)\left\{(\mu-\delta_k)\ln\left(\frac{\tau_k}{\sigma_k}-1\right)-(\mu+\delta_k)\ln\left(1-\frac{\sigma_k}{\tau_k}\right)\right\}=\\&=N(r_k)\left\{\mu\ln\frac{\tau_k}{\sigma_k}-\delta_k\ln\frac{(\tau_k-\sigma_k)^2}{\tau_k\sigma_k}\right\}=N(r_k)\{o(1)-2\delta_k\ln(\alpha_k-\beta_k)\}\\&=-N(r_k)\{2\delta_k\ln\alpha_k+o(1)\}=o(N(r_k)).\end{aligned} \tag{5.11}$$

By virtue of (1.1) with S(r) = N(r), we obtain

$$n\left(\frac{r}{2\eta_k}\right)\ln\left(1-\frac{r}{r_k/(2\eta_k)}\right) \geqslant -(1+o(1))\,n\left(\frac{r}{2\eta_k}\right)\frac{r}{r_k/(2\eta_k)} \geqslant$$
$$\geqslant -(1+o(1))\frac{N(r_k/\eta_k)}{\ln 2}\frac{r}{r_k/(2\eta_k)} \tag{5.12}$$
$$\geqslant -C\eta_k^{1-\mu}N(r_k) = o(N(r_k)).$$

We combine the expressions (5.10)-(5.12) and we integrate again by parts:

$$\ln L(r) \geqslant \left(\int_0^{r_k/\tau_k} + \int_{r_k\tau_k}^{r_k/(2\eta_k)}\right)\frac{r}{r-t}\,dN(t) + \sum\nolimits_k + o(N(r_k)) \geqslant -\left(\int_0^{r_k/\tau_k} + \int_{r_k\tau_k}^{r_k/(2\eta_k)}\right)\frac{rN(t)}{(r-t)^2}\,dt +$$
$$+\frac{r}{r-r_k/\tau_k}N\left(\frac{r_k}{\tau_k}\right) + \frac{r}{r_k\tau_k - r}N(r_k\tau_k) - \frac{r}{r_k/(2\eta_k)-r}N\left(\frac{r_k}{2\eta_k}\right) + \sum\nolimits_k + o(N(r_k)). \tag{5.13}$$

Further, by virtue of (1.1) with S(r) = N(r), we have

$$\int_0^{r_k\eta_k}\frac{rN(t)}{(r-t)^2}\,dt = o(N(r_k)); \tag{5.14}$$

$$\frac{r\,N(r_k/(2\eta_k))}{r_k/(2\eta_k)-r} \leqslant C\eta_k^{1-\mu}N(r_k) = o(N(r_k)). \tag{5.15}$$

In addition, from the expression (5.5) there follows that $\sigma_k = 1 + o(\alpha_k)$ and, therefore,

$$\frac{r}{r-r_k/\tau_k} \geqslant \frac{\sigma_k}{\sigma_k-\tau_k^{-1}} = \frac{\tau_k}{\alpha_k} + o(1); \tag{5.16}$$

$$\frac{r}{r_k\tau_k - r} \geqslant \frac{\sigma_k^{-1}}{\tau_k-\sigma_k^{-1}} = \frac{1}{\alpha_k} + o(1). \tag{5.17}$$

Combining (5.13)-(5.17), we write

$$\ln L(r) \geqslant -\left(\int_{r_k\eta_k}^{r_k/\tau_k} + \int_{r_k\tau_k}^{r_k/(2\eta_k)}\right)\frac{rN(t)}{(r-t)^2}\,dt$$
$$+\frac{\tau_k N(r_k/\tau_k) + N(r_k\tau_k)}{\tau_k - 1} + \sum\nolimits_k + o(N(r_k)). \tag{5.18}$$

Let $\gamma = r/r_k \in [\sigma_k^{-1}, \sigma_k]$, $\gamma \sim 1$, $k \to \infty$. Making use of the positivity of the kernel $r/(r-t)^2$, the estimate (1.1) with S(r) = N(r), we obtain

$$\left(\int_{r_k\eta_k}^{r_k/\tau_k} + \int_{r_k\tau_k}^{r_k/(2\eta_k)}\right)\frac{rN(t)}{(r-t)^2}\,dt \leqslant$$
$$\leqslant (1+\eta_k)N(r_k)\left(\int_{r_k\eta_k}^{r_k/\tau_k} + \int_{r_k\tau_k}^{r_k/(2\eta_k)}\right)\frac{r}{(r-t)^2}\left(\frac{t}{r_k}\right)^{\mu}dt =$$
$$= \left\{(1+\eta_k)\gamma^{\mu}\left(\int_0^{1/(\tau_k\gamma)} + \int_{r_k/\gamma}^{\infty}\right)\frac{\xi^{\mu}d\xi}{(\xi-1)^2} + o(1)\right\}N(r_k). \tag{5.19}$$

Further,

$$\left(\int_0^{1/(\tau_k\gamma)} + \int_{\tau_k/\gamma}^{\infty}\right)\frac{\xi^\mu\,d\xi}{(\xi-1)^2} = \frac{\xi^\mu}{1-\xi}\Bigg|_{\tau_k/\gamma}^{1/(\tau_k\gamma)} - \mu\left(\int_0^{1/(\tau_k\gamma)} + \int_{\tau_k/\gamma}^{\infty}\right)\frac{\xi^{\mu-1}\,d\xi}{1-\xi}; \tag{5.20}$$

$$\frac{\xi^\mu}{1-\xi}\Bigg|_{\tau_k/\gamma}^{1/(\tau_k\gamma)} = \frac{\tau_k^\mu}{\tau_k-1} + \frac{\tau_k^{-\mu}}{1-\tau_k^{-1}} + S_k, \tag{5.21}$$

and, by virtue of (5.5), setting $\gamma = 1+\varepsilon_k$, $|\varepsilon_k| = O(\beta_k)$ we have

$$\begin{aligned}|S_k| &= \left|\frac{\tau_k^{-\mu}\gamma^{-\mu}}{1-\tau_k^{-1}\gamma^{-1}} - \frac{\tau_k^\mu\gamma^{-\mu}}{1-\tau_k\gamma^{-1}} - \frac{\tau_k^\mu}{\tau_k-1} - \frac{\tau_k^{-\mu}}{1-\tau_k^{-1}}\right| = \\ &= \left|\frac{1-\mu\alpha_k}{\alpha_k+\varepsilon_k} + \frac{1+\mu\alpha_k}{\alpha_k-\varepsilon_k} - \frac{1-\mu\alpha_k}{\alpha_k} - \frac{1+\mu\alpha_k}{\alpha_k} + o(1)\right| = \\ &= \left|\frac{2\alpha_k}{\alpha_k^2-\varepsilon_k^2} - \frac{2}{\alpha_k} + o(1)\right| \leqslant \frac{4\varepsilon_k^2}{\alpha_k^3} = O\left(\frac{\beta_k^2}{\alpha_k^3}\right) = o(1).\end{aligned} \tag{5.22}$$

In addition,

$$\begin{aligned}\left(\int_0^{1/(\tau_k\gamma)} + \int_{\tau_k/\gamma}^{\infty}\right)\frac{\xi^{\mu-1}\,d\xi}{1-\xi} &= \text{v. p.}\int_0^\infty \frac{\xi^{\mu-1}\,d\xi}{1-\xi} + o(1) = \\ &= \pi\mu\operatorname{ctg}\pi\mu + o(1).\end{aligned} \tag{5.23}$$

Combining (5.19)-(5.23), we obtain

$$\left(\int_{r_k\eta_k}^{r_k/\tau_k} + \int_{r_k\tau_k}^{r_k/(2\eta_k)}\right)\frac{rN(t)}{(r-t)^2}\,dt \leqslant \left(-\pi\mu\operatorname{ctg}\pi\mu + \frac{\tau_k^\mu+\tau_k^{1-\mu}}{\alpha_k} + o(1)\right)N(r_k). \tag{5.24}$$

Further,

$$N(r_k\tau_k) \geqslant N(r_k) + n(r_k)\ln\tau_k;$$
$$N(r_k/\tau_k) \geqslant N(r_k) - n(r_k)\ln\tau_k,$$

and, therefore,

$$\begin{aligned}&\frac{\tau_k N(r_k/\tau_k) + N(r_k\tau_k)}{\alpha_k} - N(r_k)\frac{\tau_k^\mu+\tau_k^{1-\mu}}{\alpha_k} \\ &\geqslant \frac{1}{\alpha_k}\{N(r_k)(\tau_k + 1 - \tau_k^\mu - \tau_k^{1-\mu})\} - n(r_k)\ln\tau_k = \\ &= \frac{1}{\alpha_k}N(r_k)(\tau_k^\mu - 1)(\tau_k^{1-\mu} - 1) - n(r_k)\ln\tau_k = o(N(r_k)).\end{aligned} \tag{5.25}$$

Now we estimate the sum Σ_k from below. With the aid of the known Valiron–Cartan estimate (see, for example [6]), we obtain

$$\sum\nolimits_k \geqslant \left(n(r_k\tau_k) - n\left(\frac{r_k}{\tau_k}\right)\right)\ln\frac{q_k}{2e\left(1+\sqrt{q_k/2}\right)},$$

for $r \notin E_k$, $|E_k| \leqslant q_k\tau_k r_k$. In this estimate we set $q_k = \beta_k/\alpha_k$. Then by virtue of (5.5) we have

$$\tau_k q_k = O(\beta_k^2/\alpha_k) = o(\beta_k) = o\left(\sigma_k - \frac{1}{\sigma_k}\right).$$

Making use of (5.7), (5.8), and (5.6),

$$\sum_k \geqslant -CN(r_k)\delta_k \ln \frac{\beta_k^2}{\alpha_k} = o(N(r_k)),\ r \notin E_k. \tag{5.26}$$

Combining the estimates (5.18), (5.25), we obtain (5.9).

Now from (5.3), integrating twice by parts and making use of (1.1) with S(r) = N(r), we have

$$\ln M(r) \leqslant \int_0^{r_k/(2\eta_k)} \frac{rN(t)}{(r+t)^2}\,dt + o(N(r_k)) \leqslant \left(\frac{\pi\mu}{\sin \pi\mu} + o(1)\right) N(r_k),\ r \in [r_k/\sigma_k,\ r_k\sigma_k]. \tag{5.27}$$

From (5.9) and (5.27) there follows the assertion of the theorem for $\mu \leq 1/2$. The case $1/2 < \mu < 1$ is considered in the same way as in the proof of Theorem 1 by the first method. Theorem 1b is proved.

6. Examples. In this section we construct examples of entire functions which show that in the assumptions of Theorem 1 the strong Pólya peaks cannot be replaced by Pólya peaks for ln M(r, f) or by Pólya peaks for T(r, f) for ($\mu > 1/2$). These examples are based on the general ideal described below.

We consider the class of "periodic" subharmonic functions, i.e., functions satisfying the condition $u \in \mathbf{C}$, $\rho > 0$, with some $R > 1$. The number R is called the period. By V(r) = V(r, u) we denote either the characteristic

$$T(r, u) = \frac{1}{2\pi}\int_0^{2\pi} u^+(re^{i\theta})\,d\theta,$$

or B(r, u). The number $\tau \in [1, R)$ is called the limit Polya peak for V(r) if

$$\max_{0<r<\infty} \frac{V(\tau r)}{r^\rho} = V(\tau). \tag{6.1}$$

The set of all limit Pólya peaks will be denoted by P(V). It is easy to see that P(V) is contained in the set

$$\{r = e^x : 0 < x < \ln R,\ d\ln V(e^x)/dx = \rho\},$$

therefore, if the function V is real analytic and $V(r) \not\equiv cr^\rho$, then the set P(V) consists only of a finite number of points.

First we construct a subharmonic function u(z) for which the set P(V) is finite and for all $\tau \in P(V)$ we have

$$A(\tau, u) = -\infty. \tag{6.2}$$

Then, making use of V. S. Azarin's theorem on the approximation of subharmonic functions,* we find an entire function f of order ρ such that we have

$$\ln|f(z)| = u(z) + o(|z|^{\rho}),\ z \to \infty,\ z \notin C_0, \tag{6.3}$$

where C_0 is some set of zero relatively linear measure. From (6.3) and the "periodicity" of the function u there follows at once that

$$\ln M(r, f) \sim B(r, u),\ T(r, f) \sim T(r, u),\ r \to \infty.$$

The Pólya peaks for $\ln M(r, f)$ and $T(r, f)$ can be easily found. Indeed, by virtue of the "periodicity" of $V(r, u)$, this has only Pólya peaks of order ρ, and they are those and only those sequences $r_n \to \infty$, $n \to \infty$, for which there exists a sequence (r_n') such that $r_n' \sim r_n, n \to \infty$, and $(r_n') \subset \bigcup_{k=0}^{\infty} \{R^k P(V)\}$. Here tE denotes the homothetic transform of the set E with respect to the origin with ratio t. We note that if $V_1(r) \sim V_2(r)$, $r \to \infty$ then the Pólya peaks of these functions coincide. Further, by virtue of (6.2) and the "periodicity" of the function u, for any sequence of Pólya peaks (r_n) for $V(r, u)$ we have

$$\limsup_{n\to\infty} \sup\left\{\frac{A(r, u)}{B(r, u)} : \sigma^{-1}r_n \leqslant r \leqslant \sigma r_n\right\} \leqslant -q(\sigma), \tag{6.4}$$

where $q(\sigma) \to +\infty$ for $\sigma \to 1$. Now, by virtue of (6.3) and (6.4), on each sequence (r_n) of Pólya peaks for $V(r, \ln|f|)$ the relation (1.4) is satisfied.

Thus, it remains to construct "periodic" subharmonic functions with the property (6.2).

Example 1. We fix ρ, $1/2 < \rho < 1$. Let

$$\frac{1}{2} < \mu < \lambda_1 < \rho = \frac{1}{2}(\lambda_1 + \lambda_2) < \lambda_2 < 1,\quad \lambda_2 - \lambda_1 < \lambda_1^2 - \mu^2$$

By [t] we denote the integer part of the number t. We define a continuous function ρ with period 4 in the following manner: $\rho(t) = \lambda_1$, if $[t] \equiv 0 \pmod 4$; $\rho(t) = \lambda_2$, if $[t] \equiv 2 \pmod 4$; $\rho(t)$ is a linear function if $[t] \equiv 1$ or $[t] \equiv = 3 \pmod 4$. We set

$$\varphi(t) = \int_0^t \rho(\tau)\,d\tau,\ -\infty < t < \infty.$$

Then $\varphi''(t) = \min(\varphi''(t-0),\ \varphi''(t+0)) \geqslant \lambda_1 - \lambda_2 > \mu^2 - \lambda_1^2$, $\varphi'(t) \geqslant \lambda_1$, $\varphi(t) = \rho t + O(1)$, $t \to \infty$. Further,

$$\varphi(t+4) - \varphi(t) = \int_0^4 \rho(\tau)\,d\tau = 4\rho. \tag{6.5}$$

Now we define the function $g(x+iy) = e^{\varphi(x)}\cos\mu y$ in the band $\{-\infty < x < \infty,\ |y| < \pi/(2\mu)\}$. It is positive in this band and subharmonic (indeed,

$$\triangle g = (\varphi''(x) + \varphi'^2(x) - \mu^2)\,g \geqslant (\mu^2 - \lambda_1^2 + \lambda_1 - \mu^2)\,g \geqslant 0).$$

*V. S. Azarin, "On rays of completely regular growth of an entire function," Mat. Sb., 79 (121), No. 4, 463-476 (1969).

We set

$$u(re^{i\theta}) = \begin{cases} g(\ln r + i\theta), & |\theta| < \pi/(2\mu); \\ 0, & \pi/(2\mu) \leqslant |\theta| \leqslant \pi. \end{cases}$$

By virtue of (6.5), this is a "periodic" subharmonic function with period $R = e^4$. It can be verified in a straightforward manner that $P(B) = P(T) = \{e^{7/2}\}$. It can be seen easily on the graph of the function $\rho(t)$.

Let $G(z, \xi)$ be the Green function for the angle $\pi/(2\mu) < \arg z < 2\pi - \pi/(2\mu)$. We consider the potential

$$u_0(z) = \sum_{k=-\infty}^{\infty} G(z, e^{4k+7/2}) e^{4k\rho}. \tag{6.6}$$

(The series converges since $G(z, \xi) = O(|\xi|^{-\mu/(2\mu-1)})$, $|\xi| \to \infty$ and $\mu/(2\mu-1) > 1 > \rho$). The function u_0 is superharmonic inside the considered angle and "periodic" there with period e^4. Now we show that the number $\varepsilon > 0$ can be selected so small that the function

$$w(z) = \begin{cases} u(z), & |\arg z| \leqslant \pi/(2\mu); \\ -\varepsilon u_0(z), & \pi/(2\mu) < |\arg z| \leqslant \pi \end{cases}$$

be subharmonic in the plane. For this we need the following known lemma* on subharmonic functions.

LEMMA 2. Let D_1, D_2 be disjoint Jordan domains, $\partial D_1 \cap \partial D_2 \supset \ell$, where ℓ is either an interval or an arc of a circumference. Let v_1, v_2 be harmonic functions of constant sign in D_1, D_2, respectively, having zero limiting values on ℓ. Then on ℓ there exist nonvanishing derivatives $\frac{\partial v_i}{\partial n}$, $i = 1, 2$ in the direction of the exterior normals. Further, if $\frac{\partial v_1}{\partial n}(z) + \frac{\partial v_2}{\partial n}(z) \leqslant 0$ for $z \in \ell$, then the function $u(z) = v_i(z)$, $z \in D_i$, extended by zero on ℓ, is subharmonic in the domain

Making use of this lemma, we select $\varepsilon > 0$ so small that the function w be subharmonic in the neighborhood of the segments $\{\arg z = \pm\pi/(2\mu), 1 \leqslant |z| \leqslant e^4\}$. By virtue of the "periodicity" of the function w, this function is subharmonic in $\mathbb{C} \setminus \{0\}$. Finally, by the theorem on removable singularities, the function w is subharmonic in $\mathbb{C}$. The fact that property (6.2) holds follows from the construction.

Example 2. We fix ρ, $0 < \rho < 1$. We set

$$B(r) = \sum_{k=-\infty}^{\infty} \left(\frac{3}{2}\right)^{k\rho} \ln\left(1 + \left(\frac{2}{3}\right)^k r\right), \quad r > 0.$$

It is easy to see that the series converges uniformly on each compactum to a real analytic function on $(0, \infty)$ and that $B(r) \not\equiv cr^\rho$. (For example, since $B'(+0) \neq \infty$). In addition,

*M. A. Evgrafov, Asymptotic Estimates and Entire Functions [in Russian], 3rd edn., Nauka, Moscow (1979).

B(r) satisfies the "periodicity" condition with period R = 3/2. Therefore, as mentioned above, the function B has only a finite set of limit Pólya peaks $P(B) = \{r_1, r_2, \ldots, r_n\}$. We set

$$v(z) = \begin{cases} \ln|1+z|, & |1+z| \geqslant 1; \\ \varepsilon \sum_{k=1}^{n} \ln\left|\frac{z+r_k}{z+r_k-zr_k}\right|, & |1+z| < 1. \end{cases}$$

By virtue of Lemma 2, the number ε > 0 can be selected so small that the function v be subharmonic in $\mathbb{C}$. We consider the function

$$u(z) = \sum_{k=-\infty}^{\infty} \left(\frac{3}{2}\right)^{k\rho} v\left(\left(\frac{2}{3}\right)^k z\right).$$

The series converges uniformly on compacta and, therefore, u(z) is a periodic subharmonic function with period R = 3/2. Further, B(r, u) ≡ B(r), 0 < r < ∞, and from the construction there follows in a straightforward manner that (6.2) holds.

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UNIVERSAL MODELS OF LINEAR OPERATORS WITH PRESCRIBED RESTRICTIONS ON THE GROWTH OF THE RESOLVENT

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By the methods of semigroup theory one describes those classes of bounded linear operators which can be realized on the invariant subspaces of the operator $i\int_x^l f(t)\,dt$ in $L^2(0, \ell) \times \ell_r$ and of its fractional powers. The methods of construction are based on results of the type of the Paley—Wiener theorems.

By the methods of semigorup theory we construct universal models for the representation of dissipative operators with spectrum at zero and with given constraints on the growth of the resolvent.

I. We consider the dissipative local colligation [1, 2]

$$\Delta = (A, H, \varphi, E, I_E), \tag{1}$$

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