# VALIRON DEFICIENCIES OF ENTIRE CHARACTERISTIC FUNCTIONS OF FINITE ORDER

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A well-known theorem of Marcinkiewicz [1, p. 59] states that if an entire characteristic function f of finite order $\rho$ has zero Borel exceptional value, then $\rho \leq 2$. It has been proved in [2] that this theorem remains valid if it is assumed that the function f has zero exceptional value in the sense of Nevanlinna and $\delta(0, f)$ is near to 1. We will show that this theorem has no analogue for exceptional values in the sense of Valiron.

Let $H_\rho$, $2 < \rho < \infty$, be the set of entire characteristic functions f having the following properties:

$$f(z) = f(-z), \quad z \in \mathbf{C}, \tag{1}$$

$$\ln|f(iy)| \leqslant |y|^\rho + 1, \quad y \in \mathbf{R}. \tag{2}$$

The property (1) is equivalent to $F(x) = 1 - F(-x + 0)$, where F is the distribution function corresponding to f. It follows from Levi's theorem [1, p. 14] that $H_\rho$ is a topologically complete space in the compact-open topology. It follows from Baire's theorem [3] that every residual set (i.e., a countable intersection of dense open sets) is nonempty.

THEOREM 1. The set of functions of $H_\rho$ of order $\rho$ is residual.

THEOREM 2. The set of functions $f \in H_\rho$ such that $\Delta(0, f) = 1$ is residual.

COROLLARY. Let $\rho \geq 2$. There exists an entire characteristic function of order $\rho$ such that $\Delta(0, f) = 1$.

Proof of Theorem 1. Let us consider the sets

$$E_n = \left\{ f \in H_\rho : (\forall r > 0)\left[(r \geqslant n) \Rightarrow \left(\ln \ln M(r, f) \leqslant \left(\rho - \frac{1}{n}\right)\ln r\right)\right]\right\}, \quad n \in \mathbf{N}.$$

If $f \in H_\rho \setminus \left(\bigcup_{n=1}^{\infty} E_n\right)$, then f has order $\rho$. Obviously, $E_n$ are closed. We will show that $E_n$ do not contain interior points in $H_\rho$. Let us consider a neighborhood of f, i.e., the set of functions h such that $|f - h| < \varepsilon$ on some compactum $K \subset \mathbf{C}$. Let F be the distribution function corresponding to f. For each point of continuity A of the function F, $0 < A < \operatorname{rext} F$ (see [1, p. 52] for the definition of rext), let us consider the distribution function $F_1(t)$:

$$F_1(t) = \begin{cases} 0, & t \leqslant -A, \\ F(t), & -A < t \leqslant A, \\ 1, & t > A. \end{cases}$$

Let $f_1$ denote the characteristic function of the function $F_1$. It is easily seen that $f_1 \to f$ uniformly on compacta as $A \to \operatorname{rext} F$. Let us fix A such that

$$|f_1(z) - f(z)| < \frac{\varepsilon}{2}, \quad z \in K. \tag{3}$$

For every $y \in \mathbf{R} \setminus \{0\}$, by virtue of (1), we have

$$f(iy) - f_1(iy) = 2\int_A^\infty \operatorname{ch}(yt)\,dF(t) - 2\operatorname{ch}(yA)F(-A) = -2\int_A^\infty \operatorname{ch}(yt)\,d(1 - F(t)) - 2\operatorname{ch}(yA)F(-A)$$

$$= 2y\int_A^\infty \operatorname{sh}(yt)(1 - F(t))\,dt \geqslant 2y\operatorname{sh}(yA)\int_A^\infty (1 - F(t))\,dt > 0.$$

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The last expression tends to infinity as $|y| \to \infty$. Hence it follows easily that for some $\delta > 0$ the inequality

$$|f_1(iy)| \leqslant \exp(|y|^\rho + 1) - \delta \tag{4}$$

is valid for all $y \in R$. Let g be an even characteristic function of order $\rho$ with magnitude of the type 1/2. We have

$$|f_1(iy)\,g(iy)| < \exp(|y|^\rho + 1) \tag{5}$$

for $|y| > y_0 > 0$. Since $g(0) = 1$, we can choose $c > 0$ so small that the following inequalities are fulfilled:

$$|g(icy_0)| < \exp(y_0^\rho + 1)/(\exp(y_0^\rho + 1) - \delta), \tag{6}$$

$$|f_1(z)\,g(cz) - f_1(z)| < \frac{\varepsilon}{2}, \quad z \in K. \tag{7}$$

It follows from (4) and (6) that for $y \leq y_0$ we have

$$|g(icy)\,f_1(iy)| \leqslant |g(icy_0)\,f_1(iy)| \leqslant \exp(|y|^\rho + 1).$$

Together with (5) this gives $h(z) = f_1(z)g(cz) \in H_\rho$, since $f_1$ and g are even.

We now observe that $h \in H_\rho \setminus E_n$ for every n, which proves the theorem since it follows from (3) and (7) that

$$|h(z) - f(z)| < \varepsilon, \quad z \in K.$$

Proof of Theorem 2. Let us set $N_1(r, f) = \int_1^r \frac{n(t, 0, f)}{t}\,dt$ and consider the sets

$$E_n' = \left\{ f \in H_\rho : (\forall r > 0)\left[ (r \geqslant n) \Rightarrow \left(N_1(r, f) \geqslant \frac{1}{n} T(r, f)\right)\right]\right\}, \quad n \in \mathbf{N}.$$

If $f \in H_\rho \setminus \left( \bigcup_{n=1}^{\infty} E_n' \right)$, then $\Delta(0, f) = 1$ since

$$N_1(r, f) = N(r, 0, f) + O(1), \quad r \to \infty.$$

Let $J(z) \equiv 1$, $J \in H_\rho$. We know [4] that it follows from $f_j \to f \neq J$, where $f_j$ are arbitrary entire functions, that $N_1(r, f_j) \to N_1(r, f)$ and $T(r, f_j) \to T(r, f)$. Therefore, the sets $E_n'$ are closed. We will show that they do not contain interior points. Let $f \in E_n'$ and let us be given a compactum $K \subset \mathbf{C}$ and $\varepsilon > 0$. Let us consider the function $h(z) = f_1(z) \exp(-cz^2)$, where $f_1 \in H_\rho$ and $c > 0$ are defined as in the proof of Theorem 1 [the role of g is played by $\exp(-z^2)$]. It is easily seen that $\Delta(0, h) = 1$. Therefore $h \in H_\rho \setminus E_n'$ for arbitrary n. Repeating the reasonings of the proof of Theorem 1, we get

$$|f(z) - h(z)| < \varepsilon, \quad z \in K,$$

which was required to be proved.

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