Regular interpolational families are, for example, certain scales of spaces of power series (Riesz centers) or generalized spaces of power series. From what has been said there follows at once

THEOREM 3. In a regular interpolational scale of Köthe spaces all the absolute bases of pairs of spaces can be extended into any intermediate space and, moreover, are jointly quasiequivalent.

We note that the regularity of the scale is not necessary for the joint quasiequivalence of the bases [6].

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VALIRON EXCEPTIONAL VALUES OF ENTIRE FUNCTIONS OF COMPLETELY REGULAR GROWTH

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We make use of the standard notations of the theory of meromorphic functions [1]. A number $a \in \boldsymbol{C}$ is said to be an exceptional value in the Valiron sense of a meromorphic function f if

$$\Delta(a, f) = \limsup_{r \to \infty} \frac{m(r, a, f)}{T(r, f)} > 0.$$

The set of these a is denoted by $E_V(f)$. A. Hyllengren [2] has given the following description of the set $E_V(f)$ for a meromorphic function of finite order. We shall say that a set $E \subset \boldsymbol{C}$ is an H-set if there exist a sequence $(a_k) \subset \boldsymbol{C}$ and a number $\eta > 1$ such that every point $a \in E$ is contained in an infinite set of circles $|z - a_k| < \exp(-\delta_k)$ (1) and, moreover, $\delta_{k+1}/\delta_k = \eta$. A. Hyllengren's result consists in the fact that for every meromorphic function f of finite order the set $\{a \in \boldsymbol{C} : \Delta(a, f) > x\}$ is an H-set for each x, $0 < x < 1$. On the other hand, for each H-set E there exist an entire function f of finite order and a number $x > 0$, such that $\Delta(a, f) > x$ for $a \in E$.

For an important subclass of entire functions, namely the functions of completely regular growth (c.r.g.) in the B. Ya. Levin–A. Pfluger sense, the structure of the set $E_V(f)$ has

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been investigated in [3, 4]. In these investigations it is proved that if f is an entire function of c.r.g., then $Ev(f)$ contains the set of those $a\in \mathbf{C}$, for which $f-a$ is not a function of c.r.g., and can differ from it by at most a finite number of points. In [3, 4] one has constructed examples of entire functions of c.r.g. for which $Ev(f)$ has the power of the continuum.

This paper is devoted to the further investigation of the set $Ev(f)$ for entire functions f of c.r.g. We consider also the wider class Reg$^+$ of functions. This class consists of entire functions having c.r.g. on those rays where the indicator is nonnegative. We note that in the definitions of the functions of c.r.g. and of the functions of the class Reg$^+$ there occurs a proximate order $\rho(r)$. Everywhere in the sequel we shall assume that $\rho(r)\equiv\rho=$ const. The consideration of the general case does not involve additional difficulties. The functions of the class Reg$^+$ of order $\leqslant 1/2$ cannot have Valiron exceptional values. Therefore, we shall restrict ourselves to functions of order >1/2.

THEOREM 1. Let $f\in$ Reg$^+$. If $a\notin Ev(f)$, then the function f − a is of c.r.g.

This theorem is a variant of Theorem 1 from [4]; however, we give its proof for the sake of completeness.

Proof. Following V. S. Azarin [5], we consider the family of subharmonic functions $u_t(z)=t^{-\rho}\log|f(tz)-a|$, where $a\notin Ev(f)$. We consider the limit set $\mathrm{Fr}[f-a]$ of this family for $t\to\infty$. The limit is understood in the sense of generalized functions. Since $f\in$ Reg$^+$, for each function $v\in \mathrm{Fr}[f-a]$ we have $v^+(re^{i\theta})=r^\rho h^+(\theta)$, where ρ is the order and h is the indicator of the function f. Since $a\notin Ev(f)$, we have $m(r,0,f-a)=o(r^\rho)$, $r\to\infty$, consequently, $v\geqslant 0$ for all $v\in\mathrm{Fr}[f-a]$. Thus, $\mathrm{Fr}[f-a]$ consists of one function $r^\rho h^+(\theta)$, i.e., $f-a$ is of c.r.g. (see [5]); this is what we intended to prove.

A set $E\subset \mathbf{C}$ will be said to be an H_0-set if there exists a sequence of circles (1) such that each point $a\in E$ belongs to an infinite set of circles and $\delta_{k+1}/\delta_k\to\infty$, $k\to\infty$. It is easy to see that a finite union of H_0-sets is again an H_0-set. Such sets have been used by D. Drasin [6] and his method allows us to obtain the following results.

THEOREM 2. Let $f\in$ Reg$^+$. Then $Ev(f)$ is an H_0-set.

THEOREM 3. Let E be an H_0-set. Then there exist an entire function $f\in$ Reg$^+$ and a number $x>0$ such that $\Delta(a,f)>x$ for all $a\in E$.

Proof of Theorem 2. Let I be a closed arc on the unit circumference such that $h(\theta)<0$, $\theta\in I$, h being the indicator of the function f. We set

$$m(r,\ a,\ I)=\int_I \log^+|f(re^{i\theta})-a|^{-1}\,d\theta.$$

Theorem 2 follows from the following statement: the set

$$E(I)=\{a\in \mathbf{C}:\limsup_{r\to\infty} r^{-\rho}m(r,\ a,\ I)>0\} \tag{2}$$

is an H_0-set. We prove this statement. According to a known theorem on the indicator, there exists a function $\varphi(r)\downarrow 0$, $r\to\infty$ such that $\log|f(re^{i\theta})|<\varphi(r)r^\rho$, $\theta\in I$ (3). We shall assume that

φ decreases so slow that the right-hand side of (3) increases monotonically to ∞ and, in addition, $\varphi(r) > (\log r)^{-1}$ (4). We set $D(r, R) = \{z : r < |z| < rR, \arg z \in I\}$; $I(r) = \{z : |z| = r, \arg z \in I\}$; $K(r, R) = \{z\ 2r < |z| < rR/2, \arg z = \theta_0\}$, where $0 < r < \infty$, $R \geqslant 5$, θ_0 is the midpoint of the arc I. Let $P(x, z)$ be the Poisson kernel of the domain $D(r, R)$ $(x \in \partial D, z \in D)$. We denote

$$\tau(R) = \inf\{P(x, z) : x \in I(r), z \in K(r, R)\}.$$

From similarity considerations it follows that this quantity does not depend on r. From Harnack's inequality we obtain that $\tau(R) > 0$, $5 \leqslant R < \infty$.

<u>LEMMA</u>. There exists a function $R(r) \geqslant 5$, $R(r) \uparrow \infty$ such that

$$\tau(R(r)) - 2^{\rho+1}(R(r))^{\rho}\sqrt{\varphi(2rR(r))} > (\log r)^{-1}; \tag{5}$$

$$R(r) \leqslant \log r + 5, \quad r \geqslant r_0. \tag{6}$$

<u>Proof</u>. We set $r_0 = \min\{r \geqslant 1 : g(5, r) \geqslant (\log r)^{-1}\}$, where $g(R, r) = \tau(R) - 2^{\rho+1}R^{\rho}\sqrt{\varphi(2rR)}$. Assume that we have already selected r_n, $n \geqslant 0$. For r_{n+1} we select the number $\min\{r \geqslant r_n + e^n : g(n+6, r) \geqslant (\log r)^{-1}\}$. Such a choice is possible since $\varphi(r) \to 0$, $r \to \infty$. Setting $R(r) = n + 5$ for $r_n \leqslant r < r_{n+1}$, we obtain the required function.

For each $a \in E(I)$ we consider the set

$$\Gamma(a) = \{r : r^{-\rho} m(r, a, I) \geqslant \sqrt{\varphi(2rR(r))}\}. \tag{7}$$

This set is unbounded by virtue of (2). We select a number $r_0(a) \geqslant r_0$ (r_0 is from the lemma) so that we have

$$\max(\log 2, \log|a|) \leqslant \varphi(r) r^{\rho}, \quad r \geqslant r_0(a). \tag{8}$$

This is possible by virtue of the assumption that $\varphi(r) r^{\rho} \to \infty$. Let $r \in \Gamma(a) \cap [r_0(a), \infty) = \Gamma_0(a)$. We estimate the subharmonic function $\log|f(z) - a|$ on $K(r, R(r))$, making use of (3), (8), (7), (5), (4):

$$\begin{gathered} \log|f(z) - a| \leqslant \max_{z \in \partial D(r, R(r))} \log|f(z) - a| - \\ - \int_{I(r)} P(x, z) \log^{+}\frac{1}{|f(x) - a|}dx \leqslant 2\varphi(2rR(r))(2rR(r))^{\rho} - \\ - \tau(R(r)) m(r, a, I) \leqslant \{2^{\rho+1}\varphi(2rR(r))(R(r))^{\rho} - \\ - \tau(R(r))\sqrt{\varphi(2rR(r))}\} r^{\rho} \leqslant -\sqrt{\varphi(2rR(r))}(\log r)^{-1} r^{\rho} \leqslant -(\log r)^{-2} r^{\rho}. \end{gathered} \tag{9}$$

Let $r_1 \in \Gamma_0(a)$, $r_2 \in \Gamma_0(b)$, $K(r_1, R(r_1)) \cap K(r_2, R(r_2)) \neq \varnothing$, $z_0 \in K(r_1, R(r_1)) \cap K(r_2, R(r_2))$. Taking into account (9), we obtain $|b - a| \leqslant |f(z_0) - b| + |f(z_0) - a| \leqslant \exp(-r_1^{1/2})$. (10)

Now we consider the sequence (t_m), $t_1 = r_0$, $t_{m+1} = (1/4) t_m R(t_m)$. We set $J_m = [t_m, t_{m+1}]$. If for some $a \in E(I)$ we have $J_m \cap \Gamma_0(a) \neq \varnothing$, then we select one of these a's and denote it by a_m. If no such a exists, then we set $a_m = 0$. Assume now that $b \in E(I)$. The set $\Gamma_0(b)$ intersects an infinite number of segments J_m. In each of these segments there exist a point $r_m \in \Gamma_0(a_m)$ and a point $r^* \in \Gamma_0(b)$. It is easy to see that $K(r_m, R(r_m)) \cap K(r^*, R(r^*)) \neq \varnothing$, and, therefore, by virtue of (10), we have $|b - a_m| \leqslant \exp(-t_m^{1/2}) = \exp(-\sigma_m)$ (11), where $\sigma_{m+1}/\sigma_m \to \infty$. The theorem is proved.

Theorem 3 will be derived from a result of D. Drasin [6]. Let f be an entire function of order $\rho < \infty$ of normal type. We assume that the indicator of the function f is nonpositive on the segment I and that the number θ_0 lies inside I. We denote by $n(r, a, \varepsilon)$ the number of a-points of the function f in the sector $\{z: |z| \leqslant r, \theta_0 - \varepsilon \leqslant \arg z \leqslant \theta_0 + \varepsilon\}$. We say that a value $a \in C$ is maximally assumed in the neighborhood of θ_0, if there exists some $\eta > 0$ such that for all $\varepsilon > 0$ we have

$$\limsup_{r \to \infty} r^{-\rho} n(r, a, \varepsilon) \geqslant \eta. \tag{12}$$

D. Drasin has proved in [6] that the set of values, maximally assumed in the neighborhood of a ray, is an H_0-set. Further, for each H_0-set E one has constructed an entire function g of order ρ, $1/2 < \rho < 1$ such that $\log M(r, g) \sim r^\rho$, $r \to \infty$ (13) and for the indicator of this function g we have $h(\theta) \leqslant 0$ for $|\theta - \pi| \leqslant \pi\left(1 - \frac{1}{2\rho}\right)$ (14), and all the values from E are maximally assumed by the function g in the neighborhood of $\theta_0 = \pi$.

Proof of Theorem 3. Let g be the function described above. First we show that $\Delta(a, g) > x$ for all $a \in E$ and some $x > 0$. We consider the family of subharmonic functions $u_t(z) = t^{-\rho} \log|g(tz) - a|$, where $a \in E$ is fixed. We fix a sufficiently small $\varepsilon > 0$ and with the aid of (12) we find a sequence $t_k \to \infty$ such that

$$t_k^{-\rho} n(t_k, a, \varepsilon) > \eta/2 > 0. \tag{15}$$

The family $\{u_t\}$ is precompact [5] and, therefore, selecting, if necessary, a subsequence, we shall assume that $u_{t_k} \to u$ for $k \to \infty$. By virtue of (14) we have

$$u(re^{i\theta}) \leqslant 0, \quad |\theta - \pi| \leqslant \pi\left(1 - \frac{1}{2\rho}\right).$$

We set $\alpha = e^{-1}(\eta/4)^{1/\rho}$. We have $n(\alpha t_k, a, \varepsilon) \leqslant N(e\alpha t_k, a, \varepsilon) \leqslant N(e\alpha t_k, a) \leqslant T(e\alpha t_k, g) \leqslant \log M(e\alpha t_k, g) \leqslant (\eta/4) t_k^\rho$ (16). Further, it it known, that the Riesz measure of the function u_{t_k} converges weakly to the Riesz measure μ of the function u. Consequently, by virtue of (15), (16) we have $\mu(K) > \eta/4$, where by K we have denoted the sector $\{z: \alpha \leqslant |z| \leqslant 1, |\arg z - \pi| \leqslant \varepsilon\}$. Consequently, $u(z) \leqslant -x < 0$ in K, where x depends only on η and ρ. From here there follows that for any $\varepsilon > 0$ we have

$$\limsup_{r \to \infty} r^{-\rho} \int_{\pi - \varepsilon}^{\pi + \varepsilon} \log^+ |g(re^{i\theta}) - a|^{-1} d\theta > 0. \tag{17}$$

Let h be the indicator of g. We select a function g_1 of c.r.g., of order ρ and of normal type with indicator $h_1(\theta) = c \cos \rho\theta$, $-\pi \leqslant \theta \leqslant \pi$, where c is so large that $h(\theta) < c \cos \rho\theta$, $|\theta| < \pi/2\rho$. Obviously, the function $f = g + g_1 \in \mathrm{Reg}^+$. From (17) there follows that $E \subset E_V(f)$, which is what we intended to prove.

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GENERAL FORM OF EXCEPTIONAL BERNSHTEIN OPERATORS

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A quadratic mapping $V: x_j' = \sum_{i,k=1}^{n} a_{ik,j} x_i x_k (j = 1, \ldots, n)$ of the space R^n into itself is said to be a Bernshtein (or a stationary) operator if the identities $s(Vx) = s^2(x)$, $V^2x = s^2(x)Vx$ (1) hold, where

$$s(x) = \sum_{i=1}^{n} x_i.$$

The investigation of such mappings is stimulated by their role in the foundations of population genetics (see [1], Chaps. 4, 5).

By virtue of (1), the "unit" hyperplane $H = \{x | s(x) = 1\}$ is invariant for V and $V^2 = V$ on H. Consequently, all the points of the set $F_V = \mathrm{Im}(V|_H)$ are fixed for V. Conversely, if $Vx = x$ and $x \neq 0$, then by virtue of (1) we have $s(x) = 1$, i.e., $x \in H$, then $x \in F_V$. Thus, F_V coincides with the set of nonzero fixed points of the operator V. We note that the set F_V is connected (as the image of the connected set H under the continuous mapping V).

Let $x \in F_V$. Then the linear operator $L_x = V_x'$ (the derivative of the operator V at the point x).satisfies the equation $L_x^2 = L_x$, i.e., it is a projection. At the same time, L_x is a continuous function of x. Consequently, the integers $m = \mathrm{rank}\, L_x$ and $\delta = \mathrm{def}\, L_x$ do not depend on x. The pair (m, δ) is called the type of the operator V. We recall now that for each smooth mapping $\Phi: R^n \to R^n$ the maximum with respect to x of the rank of the derivative Φ_x' is called the functional rank and it is denoted by $\mathrm{rank}_f \Phi$. At the same time one can introduce the linear rank $\mathrm{rank}_l \Phi$ as the dimension of the linear hull of the set $\mathrm{Im}\, \Phi$. Obviously, $\mathrm{rank}\, g_f \Phi \leqslant \mathrm{rank}_l \Phi$.

<u>LEMMA.</u> [1, p. 77]. If V is a Bernshtein operator, then $\mathrm{rank}_f V = m$.

A Bernshtein operator V is said to be exceptional (or quasilinear) if $\mathrm{rank}_l V = m$. Dually, this means that $\dim N_V = \delta$, where N_V is the space of vanishing linear forms, i.e., the linear forms annihilating $\mathrm{Im}\, V$ (in general, $\dim N_V \leqslant \delta$).

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