4.4. Periodic points. The results in this subsection are due to Pommerenke [121].

THEOREM 2.26. *Let α be a repelling periodic point of the Blaschke product $B = \psi^{-1} \circ f \circ \psi$. Then the nontangential limit $\beta = \psi(\alpha)$ exists and is a repelling periodic point of f.*

COROLLARY. *The repelling periodic points are dense on the boundaries of simply connected Schröder domains, Boettcher domains, and Leau domains.*

THEOREM 2.27. *Suppose that $\beta \in \partial D$ is a repelling periodic point of a function f. Then the set of periodic points of B at which ψ has nontangential limit equal to β is finite (it is not known whether this set can be empty).*

§5. Critically finite endomorphisms

This section is devoted to rational endomorphisms with the simplest behavior for orbits of critical points: these orbits are absorbed by cycles. We study the theory of Thurston, following [77] and [142].

5.1. Orbifolds (see [142]). A two-dimensional *orbifold* $\mathscr{O} = (S, \nu)$ (of finite type) is defined to be a compact Riemann surface S with finitely many marked points $\{x_j\}$, to which are assigned certain weights $\nu(x_j) \in \mathbf{N} \cup \{\infty\}$, $\nu(x_j) \geq 2$. The weight function ν can be assumed to be given on the whole surface S by setting $\nu(x) = 1$ away from the marked points. Manifolds can be regarded as orbifolds with weight function $\nu = \mathbf{1}$. Our notation will correspond to this convention: $S = (S, \mathbf{1})$. We use $S^{\#}$ to denote $S \setminus \{x : \nu(x) = \infty\}$.

The *Euler characteristic* of an orbifold $\mathscr{O} = (s, \nu)$ is defined to be

$$\chi(\mathscr{O}) = \chi(S) - \sum_{x \in S} \left(1 - \frac{1}{\nu(x)}\right).$$

An orbifold is defined to be *hyperbolic* (*parabolic, elliptic*) if $\chi(\mathscr{O}) < 0$ (respectively, $\chi(\mathscr{O}) = 0$, $\chi(\mathscr{O}) > 0$).

A *cover of orbifolds* $(S_1, \nu_1) \to (S_2, \nu_2)$ is defined to be a branched cover $f: S_1^{\#} \to S_2^{\#}$ such that $\nu_2(fx) = \nu_1(x) \deg_x f$ for $x \in S_1^{\#}$. For covers of orbifolds $\mathscr{O}_1 \to \mathscr{O}_2$ the Riemann-Hurwitz formula has the simplest possible appearance:

$$\chi(\mathscr{O}_1) = \deg f \cdot \chi(\mathscr{O}_2).$$

For an arbitrary hyperbolic (parabolic) orbifold $\mathscr{O} = (S, \nu)$ there exists a *universal cover* $V \to \mathscr{O}$, where V is the hyperbolic (respectively, Euclidean) plane. This gives a representation $S^{\#} = V \setminus \Gamma$, where Γ is a group of motions of V, that gives rise to a hyperbolic (Euclidean) metric on $S^{\#}$ with conical singularities at the marked points. We call this metric the *natural* metric.

5.2. Lifting f^{-1} to the universal covering. Let $f: \overline{\mathbf{C}} \to \overline{\mathbf{C}}$ be a branched cover, and C_f the set of branch points of f.

The mapping f is said to be *critically finite* if the invariant set $P_f = \bigcup_{n=1}^{\infty} f^n C_f$ is finite. Let $\nu_f(x)$ be the least common multiple of the degrees $\deg_z f$ over all $k \in \mathbf{N}$ and $z \in f^{-k}x$. The function ν_f is the smallest function ν such that $\nu(fz)$ is a multiple of $\nu(z) \deg_z f$ for all $z \in \mathbf{C}$. If z belongs to the cycle of a branch point, then $\nu_f(z) = \infty$.

With each critically finite mapping $f: \overline{\mathbf{C}} \to \overline{\mathbf{C}}$ we associate the orbifold $\mathscr{O}_f = (\overline{\mathbf{C}}, \nu_f)$. Then the multivalued mapping f^{-1} can be lifted to a single-valued mapping of the universal covering, and we get the following result:

THEOREM 2.28 [142]. *Let f be a critically finite rational endomorphism. Then there exists a regular branched cover $P\colon V \to \overline{\mathbf{C}}^{\#}$, where V is the hyperbolic or Euclidean plane, and an analytic transformation $g\colon V \to V$ such that the diagram*

$$\begin{array}{ccc} V & \xleftarrow{\ g\ } & V \\ {\scriptstyle P}\big\downarrow & & {\scriptstyle P}\big\downarrow \\ \overline{\mathbf{C}}^{\#} & \xleftarrow[f^{-1}]{} & \overline{\mathbf{C}}^{\#} \end{array}$$

commutes. The transformation g is invertible if and only if $V = \mathbf{C}$. In this case $g^{-1}\colon z \mapsto kz$, where $|k| = \sqrt{\deg f}$.

An immediate consequence of this theorem and Schwarz' lemma is

COROLLARY 1. *A critically finite rational endomorphism f is expanding with respect to the natural metric of the orbifold $\mathscr{O}_f$:*

$$\|Df(z)\| \geq \lambda > 1, \qquad z \in \overline{\mathbf{C}} \setminus f^{-1}P_f.$$

COROLLARY 2. *If f is a critically finite rational endomorphism without superattracting cycles, then $J(f) = \overline{\mathbf{C}}$.*

The last corollary can be obtained in at least two more ways: the first goes back to Fatou (see [81], pp. 60–61, and [25]), and the second is obtained from the complete description of the dynamics on $F(f)$ [136].

5.3. Combinatorial class and transformation of the Teichmüller space. Two critically finite mappings f and g are taken to be equivalent if there exist homeomorphisms θ, $\theta_1\colon (\overline{\mathbf{C}}, P_f) \to (\overline{\mathbf{C}}, P_g)$ such that $\theta \circ f = g \circ \theta_1$, and θ is isotopic to θ_1 (rel P_f). An equivalence class is called a (finite) *combinatorial class* and denoted by $[f]$. *Does a given combinatorial class contain a rational function, and if so, how many*? This is the main question answered by Thurston's theory.

First of all, f can be assumed to be a quasiregular mapping, since every combinatorial class contains such a mapping. Let T_f be the Teichmüller space of the orbifold $\mathscr{O}_f$, i.e., of the sphere with marked points. With the combinatorial class $[f]$ we associate the transformation $\tau_f\colon T_f \to T_f$ generated by the action $\mu \mapsto f^*\mu$ on conformal structures. The question formulated above can be reduced to the problem of fixed points of the transformation τ_f:

LEMMA 2.2. *There exists a natural* 1-1 *correspondence between the rational functions* (*regarded to within a conformal conjugacy*) *in the combinatorial class* $[f]$ *and the fixed points of τ_f.*

The transformation τ_f is contracting in the Teichmüller metric. This is natural, because the latter coincides with the Kobayashi metric on T_f [129]. Actually, we can say more:

LEMMA 2.3. *If the orbifold $\mathscr{O}_f$ is hyperbolic, then the transformation τ_f^2 is strictly contracting in the Teichmüller metric. If $\mathscr{O}_f$ is parabolic, then τ_f is an isometry.*

This at once implies the following uniqueness theorem.

THEOREM 2.29. *If the orbifold $\mathscr{O}_f$ is hyperbolic, then the combinatorial class $[f]$ contains at most one rational function, to within conformal conjugacy.*

In particular, we get

THE THURSTON RIGIDITY THEOREM. *A critically finite rational endomorphism with hyperbolic orbifold does not admit quasiconformal deformations.*

In [76] there is a classification of critically finite polynomials with the help of certain combinatorial objects—the Hubbard trees.

We discuss the parabolic case in more detail below.

5.4. An existence criterion. A Jordan curve $\gamma \subset \overline{\mathbf{C}} \setminus P_f$ is said to be *nonperipheral* if each component of $\overline{\mathbf{C}} \setminus \gamma$ contains at least two points in P_f. A *multicurve* is defined to be a system $\Gamma = \{\gamma_1, \dots, \gamma_n\}$ consisting of disjoint nonperipheral curves γ_i that are nonhomotopic in $\overline{\mathbf{C}} \setminus P_f$. A multicurve Γ is said to be f-stable if each nonperipheral component of the inverse image $f^{-1}\gamma_i$ is homotopic in $\overline{\mathbf{C}} \setminus P_f$ to some curve γ_j. With each f-stable multicurve we associate the Thurston linear operator $f_\Gamma\colon \mathbf{R}^\Gamma \to \mathbf{R}^\Gamma$, where $\mathbf{R}^\Gamma$ is the real linear space with basis $\{\gamma_j\}_{j=1}^n$. This operator is defined on the basis as follows. Suppose that $\gamma_{i,j,k}$ are the components of the complete inverse image $f^{-1}\gamma_j$ that are homotopic to γ_i in $\overline{\mathbf{C}} \setminus P_f$, and $d_{i,j,k}$ is the degree of the mapping $f\colon \gamma_{i,j,k} \to \gamma_j$. Let

$$f_\Gamma(\gamma_j) = \sum_i \gamma_i \sum_k d_{i,j,k}^{-1}.$$

LEMMA 2.4. *For a given degree d of the function f and a given number p of points in P_f, there are only finitely many possibilities for the matrix of the Thurston operator.*

We can now formulate the main result of Thurston's theory.

THEOREM 2.30. *A critically finite branched cover $f\colon \overline{\mathbf{C}} \to \overline{\mathbf{C}}$ with hyperbolic orbifold is equivalent to a rational function if and only if the spectral radius of the Thurston operator is less than 1 for every f-stable multicurve Γ.*

Unfortunately, a practical verification of this criterion is extremely difficult.

5.5. The parabolic case. It is easy to list all the parabolic orbifolds. One of them is the torus $\mathbf{T}$ without marked points, and the others are the sphere $\overline{\mathbf{C}}$ with the following weight functions:

$$1)\ (\infty, \infty),\quad 2)\ (2, 2, \infty),\quad 3)\ (2, 4, 4),$$
$$4)\ (2, 3, 6),\quad 5)\ (3, 3, 3),\quad 6)\ (2, 2, 2, 2).$$

The Teichmüller space T_f in cases 1)–5) is a single point, and hence by Theorem 2.2 each combinatorial class contains a unique (to within conformal conjugacy) rational function. In case 6), T_f is the hyperbolic plane, and there is the theoretical possibility of nonuniqueness (here $\tau_f = \mathrm{id}$). We shall soon see that this possibility is actually realized.

Suppose that $\alpha\colon z \mapsto z + 1$, $\rho = \exp(\pi i/3)$, and $\wp(z, \tau)$ is the Weierstrass function with primitive periods 1 and τ. Corresponding to the orbifolds 1)–6) are the following groups Γ of motions and covers P:

$$
\begin{array}{ll}
1)\ \Gamma = \langle \alpha \rangle, & P(z) = \exp 2\pi i z; \\
2)\ \Gamma = \langle \alpha, z \mapsto -z \rangle, & P(z) = \cos 2\pi z; \\
3)\ \Gamma = \langle \alpha, z \mapsto iz \rangle, & P(z) = \wp^2(z, i); \\
4)\ \Gamma = \langle \alpha, z \mapsto \rho z \rangle, & P(z) = [\wp'(z, \rho^2)]^2; \\
5)\ \Gamma = \langle \alpha, z \mapsto \rho^2 z \rangle, & P(z) = \wp'(z, \rho^2); \\
6)\ \Gamma = \langle \alpha, z \mapsto z + \tau, z \mapsto -z \rangle, & P(z) = \wp(z, \tau),\ \operatorname{Im}\tau > 0.
\end{array}
$$

The meromorphic functions $P(z)$ arising in cases 1)–6) (and also the compositions $L \circ P$, where L is a Möbius transformation) will be called *Ritt functions* in honor of the author who discovered that they are distinguished among all meromorphic functions by a number of remarkable properties (see §11). In view of Theorem 2.28 each critically finite rational endomorphism with parabolic orbifold is connected with the "multiplication theorem" for some Ritt function:

$$P(az + b) = f(P(z)). \tag{2.1}$$

The rational endomorphisms corresponding to case 1) are the monomials $f: z \mapsto z^d$; here $J(f) = \mathbf{T}$. In case 2) we get Tchebycheff polynomials; $J(f)$ is a closed interval. In the remaining cases $J(f) = \overline{\mathbf{C}}$. The endomorphism $z \mapsto ((z-2)/z)^2$ serves as an example of case 3). A complete list of the possible constants a and b in the multiplication formula (2.1) is contained in [77].

Finally, we mention the promised exceptions to the uniqueness theorem for a rational function in a finite combinatorial class. They are obtained in case 6) for $a \in \mathbf{Z}$ and $b = 0$. These are none other than the examples of Lattès (see 1.2). For each $a \in \mathbf{Z}$ with $|a| \geq 2$ the whole one-parameter family $R_{a,\tau}$ is contained in a single combinatorial class. See [77] for a criterion for the existence of a rational endomorphism in other combinatorial classes of type 6).

§6. Holomorphic families of rational endomorphisms

In this section we investigate the dependence of the dynamics of a rational endomorphism on parameters. The problems under discussion go back to the work of Fatou, but are standard from the point of view of the modern theory of dynamical systems (see [38], [43]): 1) Is it true that a rational endomorphism in general position is structurally stable? 2) How is such an endomorphism constructed? The answer to the first question is positive (see 6.1–6.3). A hypothetical answer to the second is that a structurally stable endomorphism satisfies axiom A (subsection 6.4). In the conclusion we present a certain application of the theory developed to iteration algorithms.

6.1. The λ-lemmas. Let M be a simply connected analytic manifold, and let $\lambda_0 \in M$. A *holomorphic motion* of a set $A \subset \overline{\mathbf{C}}$ over M (with origin at λ_0) is defined to be a mapping $\varphi: M \times A \to \overline{\mathbf{C}}$ with the following properties:

a) The mapping $\lambda \mapsto \varphi(\lambda, a)$ is analytic in λ for each $a \in A$.

b) The mapping $\varphi_\lambda: a \mapsto \varphi(\lambda, a)$ is injective for each $\lambda \in M$.

c) $\varphi_{\lambda_0} = \mathrm{id}$.

REMARK. If M is not simply connected, then it is necessary to pass to the universal covering in the definition.

THE FIRST λ-LEMMA. a) *A holomorphic motion of a set* A *can be extended to a holomorphic motion of the closure* $\overline{A}$ ([26], [27], [115]).

b) *The mapping* $\varphi_\lambda: \overline{A} \to \overline{\mathbf{C}}$ *is quasiconformal for each* $\lambda \in M$ [115].

A sharp estimate of the dilatation K of the quasiconformal mapping φ_λ: $\overline{A} \to \overline{\mathbf{C}}$ is given in [60]. If $M = \mathbf{U}$ and $\lambda_0 = 0$, then $K \le (1 + |\lambda|)/(1 - |\lambda|)$. In the general case the answer is formulated in terms of the Kobayashi metric.

THE SECOND λ-LEMMA [141]. *A holomorphic motion of an arbitrary set can be extended to a holomorphic motion of the whole sphere over some neighborhood* $V \subset M$ *of the point* λ_0.

In the case when $M = \mathbf{U}$ and $\lambda_0 = 0$, we can take $\mathbf{U}_{1/3}$ as V [60]. It is unknown whether $V = \mathbf{U}$ works.

6.2. J**-stability.** A *holomorphic* (*analytic*) *family* of f_λ of rational endomorphisms is an analytic mapping $M \to \mathfrak{R}_d$. In other words, the degree of f_λ is constant, and its coefficients depend holomorphically on $\lambda \in M$.

Let $J_\lambda = J(f_\lambda)$. The endomorphism f_λ is said to be J-stable (in the family f_λ) if for all λ close to a λ_0 the restriction $f_\lambda | J_\lambda$ is topologically conjugate to $f_{\lambda_0} | J_{\lambda_0}$, and the conjugating map $\varphi_\lambda: J_{\lambda_0} \to J_\lambda$ depends continuously on λ. The family f_λ is said to be *J-stable* if all its elements are J-stable.

Let α be a periodic point of f_{λ_0}. Then under a perturbation of the parameter the point α is either simply perturbed, or splits into several periodic points. If all these points happen to be neutral (for all λ close to λ_0), then α is called a *stably neutral* periodic point.

Moreover, we consider the complex space $Z = \{(\lambda, c) \in M \times \overline{\mathbf{C}}: Df_\lambda(c) = 0\}$, and on it the sequence of holomorphic mappings $\psi_n: Z \to \overline{\mathbf{C}}$, $\psi_n(\lambda, c) = f_\lambda^n(c)$. In the case when the critical points c_j are single-valued functions of the parameter, the sequence $\{\psi_n\}$ can be regarded as a family of functions $f_\lambda^n(c_i(\lambda))$ of λ ($i = 1, \dots, 2d-2$; $n \in \mathbf{N}$). Such a family was first considered by Levin [21]. A family of mappings $Z \to \overline{\mathbf{C}}$ is said to be *normal* if it is precompact.

THEOREM 2.31 ([26], [27], [115]). *Suppose that* f_λ *is a holomorphic family of rational endomorphisms. Then the following conditions are equivalent:*

a) *For every* $\lambda \in M$ *the neutral periodic points of the function* f_λ *are stably neutral.*

b) *The Julia set* J_λ *moves holomorphically over* M.

c) *The family* f_λ *is* J*-stable.*

d) *The sequence* $\{\psi_n\}$ *is normal.*

In proving that a)$\Rightarrow$b) it is natural first to construct the holomorphic motion φ_λ on the repelling periodic points, and then to extend it to the whole Julia set by the first λ-lemma. This motion gives the conjugacy required for c): $f_\lambda \circ \varphi_\lambda = \varphi_\lambda \circ f_{\lambda_0}$ on J_{λ_0}.

THEOREM 2.32 ([26], [27], [115]). *The set of* J*-stable endomorphisms is open and dense in every holomorphic family.*

Thus, the manifold M has a nowhere dense subset Λ of J-unstable values of the parameter such that Λ partitions M into domains of J-stability. The set Λ has a very complicated structure: every neighborhood of a point in it intersects countably many components of the complement $M \setminus \Lambda$. The following can be said about the dynamics of the set of J-unstable functions:

THEOREM 2.33. a) *A dense subset of Λ is formed by those λ such that the orbit of one of the critical points of f_λ is absorbed by a repelling cycle* ([21], [26]).

b) *If $\lambda \in \Lambda$, then there exists a sequence $\lambda_k \to \lambda$ such that f_{λ_k} has a super-attracting cycle of order $k \geq k_0$* ([21], [26]).

c) *For those $\lambda \in \Lambda$ that are typical (residual) in the Baird sense*([4]) *the orbit of one of the critical points is contained in the Julia set J_λ and is dense there* ([26], [27]).

6.3. Structural stability. An endomorphism f_λ is said to be *structurally stable* in a holomorphic family f_λ if for all λ sufficiently close to a λ_0 the endomorphisms f_{λ_0} and f_λ are topologically conjugate on the whole sphere $\overline{\mathbf{C}}$, and the conjugating homeomorphism depends continuously on λ.

THEOREM 2.34 [115]. *The set of structurally stable endomorphisms is open and dense in every holomorphic family. A conjugacy is generated by a holomorphic motion of the sphere.*

Structural stability differs from J-stability by natural supplementary requirements. In the case when $\{f_\lambda\}$ is the whole manifold of rational functions they are formulated as follows: the critical points are nondegenerate and lie in different large orbits.

6.4. The Fatou problem. *Is it true that a rational endomorphism in general position satisfies axiom A?*

From the point of view of Theorem 2.34 the problem reads as follows: *is it true that J-stability implies axiom A?*

(The converse is true: this follows, for example, from d)⇒c) in Theorem 2.31.) The components of the set $M \setminus \Lambda$ in which f_λ satisfies axiom A will be called the *A-domains.*

Assume that with each point z of a measurable set $X \subset J$ of positive measure we associate a line L_z of the tangent plane T_z. Such an object is called a *measurable field of lines* on J (although this field is not defined everywhere on J). If the set X is invariant and $Df(L_z) = L_{fz}$, then the field of lines is said to be *invariant.* Away from ∞ and the poles of f an invariant field of lines is analytically determined by a measurable function θ on X such that $\theta(fz) = \theta(z) + \arg f'(z) \pmod{\pi}$.

THEOREM 2.35 ([115], [104]). *If a rational endomorphism is J-stable and does not have measurable invariant fields of lines on the Julia set, then it satisfies axiom A.*

Thus, the Fatou problem has been reduced to the ergodic problem of the absence of invariant measurable fields of lines on $J(f)$. We remark that the analogous ergodic problem for Kleinian groups has been solved (Sullivan [135]).

6.5. The McMullen rigidity theorem and iteration algorithms. A family f_λ will be said to be *subordinate* to a family g_μ if each function f_λ is conformally conjugate to some function g_μ. A family subordinate to a single-element family is said to be *trivial.*

([4])That is, the set of such λ is the complement of a set of first category in Λ.

A manifold M is said to be a *Liouville* manifold if all the bounded holomorphic functions on M are constant (for example, Riemann surfaces of finite type are such manifolds).

THEOREM 2.36 [107]. *A J-stable holomorphic family $\{f_\lambda\}_{\lambda\in M}$ parametrized by a Liouville manifold M is either trivial or subordinate to the one-parameter family of Lattès* (*see* 1.2).

Using Theorem 2.36, McMullen investigated iterative algorithms for solving algebraic equations. An *iterative algorithm* is defined to be a rational mapping $p \mapsto f_p \in \mathfrak{R}_d$, where p is a polynomial of degree m, such that the iterates f_p^n converge to roots of the polynomial p on an open dense subset of $\mathbf{C}$. For $m = 2$ the Newton iterative process is such an algorithm. The next result gives a solution of a problem of Smale [134].

THEOREM 2.37 [134]. *There are no iterative algorithms for polynomials of degree $m \geq 4$. For $m = 2$ and 3 there are such algorithms, and they admit a complete description.*

§7. The Mandelbrot set

In this section we consider a one-parameter family of quadratic polynomials $f_c\colon z \mapsto z^2 + c$ $(c \in \mathbf{C})$. Despite the elementary nature of the situation, the bifurcation diagram of this family has an exceedingly rich and intricate structure (Figure 5). It was the picture of this diagram obtained by Mandelbrot with the help of a computer [111] that stimulated the great interest in the circle of problems as a whole. The theoretical investigation of the Mandelbrot set requires invoking subtle techniques of quasiconformal surgery. The main results in this direction have been obtained by Douady and Hubbard ([75], [76], [78]).

7.1. Trees of A-domains. The polynomials $f_c\colon z \mapsto z^2 + c$ are pairwise not conformally conjugate, and each quadratic polynomial is conformally conjugate to some f_c. Thus, the family f_c is the quotient of the space of quadratic polynomials with respect to the action of the affine group $z \mapsto az + b$ by conjugacies.

The only critical point of the polynomials f_c is 0. Its orbit as a function of the parameter c is given by the sequence of polynomials $\psi_n(c) = f_c^n(0)$, $\deg \psi_n = 2^{n-1}$. It is easy to see that either $|\psi_n(c)| \leq 2$, or $|\psi_n(c)| \to \infty$, $n \to \infty$. In the first case the Julia set is connected, and in the second case it is a Cantor set (Theorems 2.10 and 2.20). The Mandelbrot set is defined to be $\mathbf{M} = \{c \in \mathbf{C}\colon J(f_c) \text{ is connected}\}$.

THEOREM 2.38. a) *The Mandelbrot set is compact*;
b) *its complement* $\mathbf{C}\setminus\mathbf{M}$ *is connected*;
c) *each component of its interior* $\mathbf{M}^0$ *is simply connected*;
d) *the set of J-unstable endomorphisms f_c coincides with the boundary* $\partial\mathbf{M}$.

If an endomorphism f_c has an attracting cycle $\alpha(c)$, then it satisfies axiom A, and hence $c \in \mathbf{M}$. Let V be the component of $\mathbf{M}^0$ containing c (an A-domain). In this domain the multiplier $\lambda_V(c)$ of the cycle $\alpha(c)$ is a single-valued analytic function. On its boundary $|\lambda_V(c)| = 1$, from which it is clear that ∂V is a piecewise analytic Jordan curve, and $\lambda_V\colon V \to \mathbf{U}$ is branched

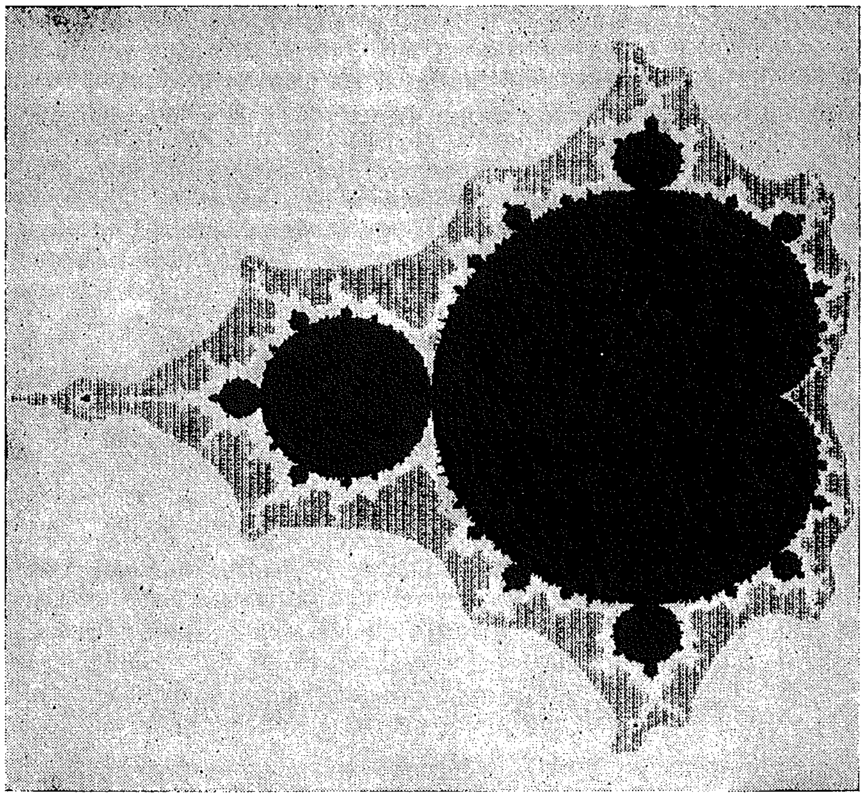

FIGURE 5. The Mandelbrot set

cover. A much more subtle fact actually holds:

THEOREM 2.39 ([75], [76]). *The multiplier* $\lambda_V \colon V \to \mathbf{U}$ *is a conformal isomorphism in each* A*-domain* V.

Consequently, the A-domain V contains a unique value of the parameter c_0 for which $\lambda(c_0) = 0$, i.e, the cycle $\alpha(c_0)$ is superattracting. The point c_0 is called the *center* of the A-domain. The boundary ∂V contains a unique point c_1 for which $\lambda(c_1) = 1$. This point is called the *root* of the A-domain.

The largest domain in Figure 5 is an A-domain with center at zero. It is bounded by a curve of cardioid type with singularity (cusp) at the point $c = 1/4$, the root of the domain. Lying in this domain are all polynomials f_c satisfying axiom A and such that the Julia set is a Jordan curve.

As the parameter c runs through the boundary ∂V, the multiplier λ_V runs around the circle $\mathbf{T}$ once. Of main interest are the values $c \in \partial V$ for which $\lambda_V(c)$ is a root of unity (of order $q > 1$). At such a point the cycle β merges with the cycle α to an order q times greater. If ∂V is regularly intersected, then these cycles again dissociate, but β becomes attracting.

Thus, other A-domains abut V at a dense set of points $c \in \partial V$. Countably many A-domains abut each of them in turn, and so on. As a result we obtain an infinite tree of domains. Such a tree D_1 is nicely visible in Figure 5, growing from the A-domain with center at zero. The Mandelbrot set actually contains not one, but countably many, such trees D_i. *Is it true that* $\bigcup D_i$ *is dense in* $\mathbf{M}$? This question, which is a variant of Fatou's problem (subsection 6.4) remains open so far. Douady and Hubbard showed that a positive answer would follow from local connectedness of the Mandelbrot set $\mathbf{M}$.

An A-domain is said to be *primitive* if it is not obtained from another A-domain by a bifurcation. In pictures it is easy to distinguish primitive A-domains from nonprimitive ones, since the former are bounded by curves of cardioid type (with cusp at the root), while the latter are smooth curves.

7.2. Uniformization of $\overline{\mathbf{C}} \setminus \mathbf{M}$.

THEOREM 2.40 ([75], [76]). *The Mandelbrot set $\mathbf{M}$ is connected.*

In other words, the complement $\overline{\mathbf{C}} \setminus \mathbf{M}$ can be mapped conformally onto the unit disk $\mathbf{U}$. This mapping can be constructed explicitly: $\Phi(c) = \varphi_c(c)$, where φ_c is the Boettcher function of the polynomial f_c for ∞. According to Carathéodory's theorem, the Mandelbrot set is locally connected if and only if the inverse function Φ^{-1} can be continuously extended to the closed disk $\mathbf{U}$. Douady and Hubbard obtained the following result in this direction.

THEOREM 2.41 [76]. *For every rational number $\theta = p/q$ the function Φ^{-1}: $\mathbf{U} \to \overline{\mathbf{C}} \setminus \mathbf{M}$ has a radial limit c_θ at the point $\exp(2\pi i\theta)$. Further:*

a) *if q is even, then the endomorphism f_{c_θ} is critically finite, since the orbit of zero is absorbed by a repelling cycle;*

b) *if q is odd, then c_θ is a root of some A-domain.*

The value of the parameter c_θ is called *a Misiurewicz point* in case a). The next theorem shows that in a neighborhood of Misiurewicz points the Mandelbrot set is similar to the corresponding Julia set. We consider the natural topology on the space of subsets of the complex plane: $X_n \to Y$ if $X_n \cap \mathbf{U}_r \to Y \cap \mathbf{U}_r$ in the Hausdorff metric([5]) for all $r > 0$, where $\mathbf{U}_r = \{z : |z| < r\}$.

THEOREM 2.42 [76]. *Suppose that $c \in \mathbf{M}$ is a Misiurewicz point, and λ is the multiplier of a cycle that absorbs the orbit of zero. Then there exists a closed set $Z \subset \mathbf{C}$ such that $\lambda Z = Z$, and*

$$\lambda^n(J(f_c) - c) \to Z,$$
$$\lambda^n(\mathbf{M} - c) \to \rho Z, \qquad n \to \infty$$

for some $\rho \in \mathbf{C}$.

The critically finite functions of the second type with 0 a periodic point correspond to the centers of A-domains. Let P_n be the set of centers of A-domains of period n (P_n consists of $\deg \psi_n = 2^{n-1}$ points). Let μ_n be the uniform probability measure on P_n. It turns out that $\mu_n \to \mu$, where μ is the equilibrium measure on the Mandelbrot set (Levin and Lyubich, 1981). Using subtle estimates of the rate of convergence, Levin [22] obtained a number of interesting arithmetic properties of the coefficients of a conformal mapping $\mathbf{U} \to \overline{\mathbf{C}} \setminus \mathbf{M}$ (see also [97]).

7.3. The monotonicity of the entropy. In conclusion we dwell on a certain problem in the theory of one-dimensional dynamical systems that was dealt with successfully by using complexification and the Teichmüller theory (Douady and Hubbard [76], Thurston [142]).

For real c the transformation f_c have the invariant circle $\overline{\mathbf{R}}$. Let h_c be the topological entropy of the restriction $f_c|\mathbf{R}$. It can be defined by

$$h_c = \lim_{n\to\infty} \frac{1}{n} \ln N_n(c),$$

([5])The distance between sets A and B in the Hausdorff metric is $\sup_{a\in A} \inf_{b\in B} \rho(a, b)$.

where $N_n(c)$ is the number of real periodic points with period n for the endomorphism f_c.

THEOREM 2.43. *The topological entropy h_c increases monotonically as c decreases.*

In fact, more can be proved: the real periodic points cannot go off into the complex plane as c decreases. The proof uses the kneading theory of Milnor and Thurston (see [65]). This theory gives us that in the contrary case there are two A-domains with real centers c_1 and c_2 such that the critically finite endomorphisms f_{c_1} and f_{c_2} are contained in a single combinatorial class. This contradicts the uniqueness Theorem 2.29.

§8. Quasiconformal deformation

The *maximal quasiconformal deformation* of an endomorphism f is defined to be the space of all rational endomorphisms that are quasiconformally conjugate to f. Let $\mathrm{qc}(f)$ be the maximal quasiconformal deformation, factored by the action of the Möbius group of conjugacies. We write $g \in \mathrm{qc}(f)$ with the understanding that the endomorphism g is considered up to conformal conjugacy. In this section we describe a parametrization of $\mathrm{qc}(f)$ by a suitable Teichmüller space.

Let S be the union of the Riemann surfaces (with marked points and possibly an action of the rotation group) associated with the cycles of the components of the Fatou set (see §2), T_S the corresponding Teichmüller space, and T_J the space of measurable invariant fields of lines on the Julia set (see 6.4). The *Teichmüller space of the function* f is defined to be $T(f) = T_S \times T_J$. The space $T(f)$ is a finite-dimensional complex analytic manifold.

The *automorphism group* $\mathrm{Aut}\, f$ of f is the group of Möbius transformations commuting with f. The *modular group* $\mathrm{Mod}\, f$ is the group of quasiconformal homeomorphisms $\overline{\mathbf{C}} \to \overline{\mathbf{C}}$ commuting with f, factored by the component of the identity. The modular group acts in a natural way on the Teichmüller space $T(f)$.

THEOREM 2.44 (SULLIVAN [140]). *The group $\mathrm{Mod}\, f$ acts on $T(f)$ in properly discontinuous fashion. The orbit space $T(f)/\mathrm{Mod}\, f$ can be identified in a natural way with $\mathrm{qc}(f)$. The isotropy group of a point $x \in T(f)$ is isomorphic to the group $\mathrm{Aut}(g)$, where $g \in \mathrm{qc}(f)$ corresponds to the orbit of x.*

This theorem gives a unified look at a broad circle of questions considered above: the absence of wandering domains, Theorem 2.39 on the multiplier, the connectedness of the Mandelbrot set, Theorem 2.35, and others (see [140] and [29] for more details).

§9. Quasiconformal surgery

In certain cases a holomorphic dynamical system can be cut into parts and then a new system can be glued together from these parts. This procedure, whose sources lie in the theory of Kleinian groups, Douady calls surgery. Surgery of rational functions arose recently ([73], [74], [109], [139]), and in this area there are many conjectures that have so far been confirmed only experimentally. We confine ourselves here to certain results that have been rigorously proved.

9.1. Polynomial-like mappings. Let U_1 and U be simply connected domains with $\overline{U}_1 \subset U$. A polynomial-like mapping is defined to be a branched cover

$f: U_1 \to U$ that is holomorphic in a neighborhood of $\overline{U}_1$. Denote by $K(f)$ the (closed) set of points z such that $f^n z \in U_1$ for $n \in \mathbf{N}$. Two mappings $f: U_1 \to U$ and $g: V_1 \to V$ are said to be *quasiconformally equivalent* if there exists a quasiconformal homeomorphism $\varphi: U' \to V'$, where U' and V' are neighborhoods of $K(f)$ and $K(g)$, respectively, such that $g \circ \varphi = \varphi \circ f$ on $f^{-1}(U')$. An equivalence is said to be a *hybrid* equivalence if $\overline{\partial}\varphi = 0$ a.e. on $K(f)$. An *external equivalence* is an analytic isomorphism $\varphi: U' \setminus K(f) \to V' \setminus K(g)$ that conjugates f and g on $f^{-1}(U') \setminus K(f)$.

THEOREM 2.51 (Douady-Hubbard [78]). *For each two polynomial-like mappings f and g of the same degree with connected sets $K(f)$ and $K(g)$ there exists a third mapping that is hybrid-equivalent to f and external-equivalent to g.*

COROLLARY 1. *An arbitrary polynomial-like mapping is hybrid-equivalent to a polynomial of the same degree.*

COROLLARY 2. *A polynomial of degree d has at most $d-1$ nonrepelling cycles.*

Indeed, it is possible to perturb a polynomial in the class of polynomial-like mappings of the same degree in such a way that all the nonrepelling cycles become attracting, and then to use Corollary 1 and Theorem 2.7 in §2.

Suppose now that $\overline{f} = \{f_\lambda\}_{\lambda \in M}$ is a holomorphic family of polynomial-like mappings (the exact definition is in [78]). If $\deg f_\lambda = 2$, then f_λ is equivalent to some polynomial $z \mapsto z^2 + c$, by Corollary 1. Let $c = \chi(\lambda)$, $\chi: M \to \mathbf{C}$. Denote by $\mathbf{M}_{\overline{f}}$ the set of values of the parameter λ for which $K(f_\lambda)$ is connected. If $\mathbf{M}_{\overline{f}}$ is compactly contained in M and certain additional conditions are satisfied, then it can be asserted that $\chi: \mathbf{M}_{\overline{f}} \to \mathbf{M}$ is a quasiconformal homeomorphism.

Suppose now that W is a primitive A-domain of order p in $\mathbf{M}$, c_0 is its center, and D_{c_0} is the Boettcher domain of $f_{c_0}: z \mapsto z^2 + c_0$ containing 0. Then there exists a neighborhood D_1 of D_{c_0} such that $f^p_{c_0}|D_1$ is a polynomial-like mapping of second degree. It turns out that this mapping can be imbedded in a holomorphic family $\overline{f}$ whose parameter varies in a neighborhood M of the set $\overline{W}$ in such a way that $\mathbf{M}_f \subset M$. This implies the well-known experimental fact that $\mathbf{M}$ *contains infinitely many homeomorphic copies of itself* (Douady-Hubbard [74], [78]).

To turn the sketch given above into a rigorous proof requires very subtle techniques developed by Douady and Hubbard.

9.2. Proof of the Shishikura theorem. The results of Shishikura (§2.6) are based on the following assertion, which is easily obtained from the measurable Riemann theorem.

SHISHIKURA'S LEMMA. *Suppose that $g: \overline{\mathbf{C}} \to \overline{\mathbf{C}}$ is a quasiregular mapping, $E_i \subset \overline{\mathbf{C}}$, $1 \le i \le m$, are disjoint open sets, and $p_i: E_i \to E_i'$ are quasiconformal mappings onto certain domains. Assume that:*

1) $g(E) \subset E$, *where* $E = \bigcup_i E_i$;
2) $p \circ g \circ p_i^{-1}$ *are holomorphic on* E_i' *(here p is the union of the p_i)*;
3) $\overline{\partial} g = 0$ *a.e. on* $\overline{\mathbf{C}} \setminus g^{-N}(E)$, *where N is a fixed positive integer.*

Then there exists a quasiconformal mapping $\varphi\colon \overline{\mathbf{C}} \to \overline{\mathbf{C}}$ *such that* $\varphi \circ g \circ \varphi^{-1}$ *is a rational endomorphism,* $p \circ \varphi^{-1}$ *is conformal on* $\varphi(E)$, *and* $\overline{\partial} g = 0$ *a.e. on* $\overline{\mathbf{C}} \setminus \bigcup g^{-n}(E)$.

We first prove the estimate $N_A + N_L + N_I + N_S \leq 2d - 2$. For this we restructure f in such a way that the neutral irrational cycles become attracting cycles, without affecting the nature of the attracting and neutral rational cycles. Let $\{\alpha_i\}$ and $\{\beta_j\}$ be the neutral irrational and neutral rational periodic points, respectively. It can be assumed that they lie in $\mathbf{C}$, and $f(\infty) = \alpha_1$. Let h be a polynomial of degree m with roots at the periodic points α_j and β_j, and assume that the β_j are roots of high multiplicity (larger than the number of Leau petals) and $h'(\alpha_i) = -1$. Denote by ρ a smooth function such that $\rho(x) = 1$ for $0 \leq x \leq 1$, and $\rho(x) = 0$ for $x \geq 2$. Let $H_\varepsilon(z) = z + \varepsilon^{m+1} h(z)\rho(\varepsilon|z|)$. Obviously, H_ε is a quasiconformal homeomorphism (for sufficiently small ε), and $H_\varepsilon \to \mathrm{id}$ as $\varepsilon \to 0$. Consider the quasiregular transformation $g_\varepsilon = f \circ H_\varepsilon$. It is holomorphic outside the set $V_\varepsilon = \{z\colon |z| \geq \varepsilon^{-1}\}$, and its periodic points have the required properties. Let D_ε be the Schröder domain of g_ε containing the point α_1, and let r_ε be the distance from α_1 to ∂D_ε. It can be shown that r_ε decreases more slowly than ε^α as $\varepsilon \to 0$ for every $\alpha > 0$. Therefore, the inclusion of $g_\varepsilon V_\varepsilon \subset D_\varepsilon$ holds for sufficiently small $\varepsilon > 0$. Use of Shishikura's lemma (with $p = \mathrm{id}$) gives a rational function f_ε such that $N_A(f_\varepsilon) \geq N_A(f) + N_S(f) + N_I(f)$ and $N_L(f_\varepsilon) = N_L(f)$. It remains to use Theorems 2.7 and 2.11.

Assume now that f has an invariant Arnol′d-Herman ring A, and consider an invariant analytic curve $\gamma \subset A$. This curve separates the sphere into two parts D_+ and D_-. Let p_+ be a quasiconformal mapping of D_- onto the disk that conjugates $f|\gamma$ to a rotation $T\colon z \mapsto \lambda z$.

Let

$$g_+(z) = \begin{cases} f(z), & z \in \overline{D}_+, \\ p_+^{-1} \circ T \circ p_+, & z \in D_-. \end{cases}$$

A transformation g_- coinciding with f on $\overline{D}_-$ is defined similarly. Shishikura's lemma is applicable to both the transformations g_+ and g_-. It is used to construct rational endomorphisms f_+ and f_- such that each has a Siegel disk and such that their sum has just as many critical points as f. As a result of this surgical operation the invariant Arnol′d-Herman ring turns into two Siegel disks. The construction leads to a proof of the Shishikura inequalities in the case when all the annuli are invariant. The general case of periodic annuli requires a certain complication in the construction.

The sharpness of the Shishikura estimates can be proved by applying inverse surgery to Example 2.4 in 2.2.

§10. The analogy with the theory of Kleinian groups

This analogy has served as a fruitful source of new ideas in both theories since the time of Fatou. We refer the reader to the book [20] for the main concepts and facts in the theory of Kleinian groups. The following (far from complete) table is borrowed mainly from Sullivan's paper [154]. A question mark means that the analogous problem is open.

TABLE 1

Finitely generated group Γ of Möbius transformations of the sphere	Rational endomorphism of the sphere
Concepts	
Set of discontinuity Ω	Fatou set F
Limit set Λ	Julia set J
Fixed points of loxodromic (parabolic) transformations	Repelling (neutral rational) periodic points
An invariant component of the set Ω	A completely invariant component of F
Γ is a Kleinian group	$F \neq \varnothing$
Γ is a Fuchsian group	f is a Blaschke product
Γ is a quasi-Fuchsian group	J is a Jordan curve
Γ is a geometrically finite group without parabolic elements	f satisfies axiom A
Γ is a Schottky group	f satisfies Axiom A and J is a Cantor set
The Riemann surface Ω/Γ	The Riemann surface S_f
Quasiconformal deformations and associated concepts	Quasiconformal deformations and associated concepts
Theorems and problems	
Λ is perfect	J is perfect
Λ is nowhere dense or $\Lambda = \overline{\mathbf{C}}$	J is nowhere dense or $J = \overline{\mathbf{C}}$
Ω consists of $0, 1, 2,$ or countably many components	F consists of $0, 1, 2,$ or countably many components
The fixed points of loxodromic transformations are dense in Λ	The repelling periodic points are dense in J
The Ahlfors finiteness theorem	Sullivan's theorem on the absence of wandering domains
The Ahlfors problem on the measure of the limit set	Fatou's problem on the measure of the Julia set
There are no invariant measurable fields of lines on Λ	?
?	A rational endomorphism in general position is structurally stable
Structural stability $\Rightarrow \Gamma$ is geometrically finite and without parabolic elements	?

It would seem useful to imbed the two parallel theories in a single theory of semigroups of analytic transformations.

§11. Commuting functions

The papers [96] of Julia and [83] of Fatou are devoted to the problem of describing commuting rational functions by means of methods in the theory of iterates they created. Ritt [128] soon gave a solution of this problem complete in a certain sense (and by another method). We formulate the main result:

THEOREM 2.52 [128]. *Suppose that the rational functions* f *and* g *commute and do not have common iterates, and that* $\deg f \geq 2$ *and* $\deg g \geq 2$. (6) *Then* f

(6)The case when one of the functions is a Möbius function can be investigated directly.

and g are critically finite functions with a common parabolic orbifold. In other words, f and g satisfy equations (2.1) *in* §5 *with a common Ritt function.*

COROLLARY ([96], [83]). *Assume that the nonlinear polynomials f and g commute and do not have common iterates. Then they can be reduced by a single Möbius transformation to a pair of monomials z^n, z^m or a pair of Tchebycheff polynomials.*

We confine ourselves to a presentation of the approach of Julia and Fatou, which leads to a proof of the corollary. If f and g commute, then g acts on the set of fixed points of f while preserving their type. From this it follows, first, that f and g have a common Julia set J. Second, replacing f and g by certain of their iterates, we can assume that they have a common repelling fixed point α. It will be assumed without loss of generality that $\alpha = 0$. Let $\lambda_1 = f'(0)$ and $\lambda_2 = g'(0)$. We consider the Poincaré equation

$$\Phi(\lambda_1 z) = f(\Phi(z)), \qquad \Phi(0) = \alpha, \qquad \Phi'(0) = 1. \tag{2.2}$$

By Theorem 1.1, it has a unique normalized solution that is holomorphic in a neighborhood of zero and that can then be uniquely extended to a meromorphic solution on $\mathbf{C}$. Let $F(z) = g(\Phi(z))$. A direct verification shows that F also satisfies equation (2.2); further, $F(0) = 0$ and $F'(0) = \lambda_2$. Consequently, $F(z) = \Phi(\lambda_2 z)$, and hence $\Phi(\lambda_2 z) = g(\Phi(z))$.

We have obtained that Φ satisfies the two Poincaré equations with multipliers λ_1 and λ_2. This implies that $\lambda_1^n \neq \lambda_2^m$ for all m, $n \in \mathbf{Z}$ (otherwise f and g would have a common iterate). We now consider the set $G = \Phi^{-1}(J)$. It is invariant under the action of the free abelian group generated by the transformations $z \mapsto \lambda_1 z$ and $z \mapsto \lambda_2 z$. An easy consequence of this is the alternative: a) G is a finite union of logarithmic spirals; b) $G = \overline{\mathbf{C}}$. Only the first case is possible for polynomials, and in this case Theorem 2.25 concludes the proof.

It remains to describe the functions with common iterates.

THEOREM 2.53 (Ritt [126], [128]). *Suppose that f and g are a pair of polynomials with $f^n = g^m$. Then there exists a polynomial $h(z) = zh_1(z^k)$ such that f and g can be reduced simultaneously to form the $\varepsilon_1 h^s$ and $\varepsilon_2 h^l$, where ε_1 and ε_2 are kth roots of unity.*

For general rational functions the corresponding result has a considerably more complicated formulation, and Ritt himself regarded it as unsatisfactory [128]. As we have already noted, the Julia sets of commuting functions coincide. It is natural to try to prove the converse of this assertion:

THEOREM 2.54 (Baker and Eremenko, Ann. Acad. Sci. Fenn. Ser. A I Math. **12** (1987), 229–236). *Let f and g be polynomials with a common Julia set J. Assume that there do not exist Möbius transformations of finite order with respect to which J is invariant. Then f and g commute.*

The reader interested in investigating other functional equations is referred to [95], [127], [8], and [82].

In conclusion we mention the article [9] of Veselov, who gives interesting multidimensional examples of commuting transformations connected with semisimple Lie algebras.

CHAPTER 3
MEASURABLE DYNAMICS OF RATIONAL ENDOMORPHISMS

In Chapter 2 we described a very complete picture of the dynamics of rational endomorphisms on the Fatou set. The situation is different with dynamics on the Julia set: mixing and denseness of cycles give only a very general idea of the instability and chaos. It is at once clear that it is not possible to give a complete picture of the chaotic dynamics. We can hope only to describe the behavior of a trajectory that is typical in this or that sense. For example, we can specify an invariant or quasi-invariant measure on $J(f)$ and understand typicalness in the sense of this measure. This is the approach used in the present chapter. Only very natural measures are considered here: Lebesgue measure (§2), Hausdorff measure (§3), the maximal entropy measure (§4), and harmonic measure (§5). If an endomorphism satisfies axiom A, then it is relatively simple to investigate. The theory of Markov partitions and Gibbs measures is a powerful aid here. If we waive axiom A, then serious technical difficulties arise at once, and thus many results in this chapter are only first steps.

§1. Elements of the general theory of dynamical systems

The present section sketches the necessary preliminary facts from the general theory of dynamical systems. The novice cannot learn this theory from what is given, but only absorb the concepts and facts given below and consider the references to the literature cited. For a number of concepts we do not even present the definitions common in systematic courses, but instead state theorems that will actually be used as definitions. Special emphasis is given to concepts and facts specific to noninvertible transformations, because almost no attention has been given to them in the general literature, and for us they are the main object of investigation. The theory of expanding endomorphisms has important significance for us. On the other hand, it is directly applicable to rational endomorphisms satisfying axiom A. On the other hand, it also models well the situation beyond the limits of axiom A, since rational endomorphism are correctly interpreted in many problems as nonuniformly expanding mappings.

1.1. Measurable partitions of a Lebesgue space [40]. A measure space (X, μ) is called a *Lebesgue space* if it is isomorphic to a closed interval with a Lebesgue-Stieltjes measure. For example, a complete separable metric space equipped with a Borel measure(7) is a Lebesgue space (we shall deal only with such spaces below). Versatile techniques of measurable partitions and conditional measures are available for Lebesgue spaces.

We remark first of all that we regard all partitions mod 0: $\xi = \eta$ if ξ and η coincide after restriction to some set $Y \subset X$ of full measure. The element of a partition ξ containing a point x is denoted by $\xi(x)$. A partition ξ is said to be *measurable* if its elements are level sets of some measurable function $\varphi: X \to [0, 1]$. The natural order is introduced on the set of measurable partitions: $\xi \leq \eta$ if the elements of ξ are formed from elements of η. Every family $\{\xi_i\}$ of measurable partitions has a supremum $\bigvee \xi_i$ and an infimum $\bigwedge \xi_i$. The partition ε into singleton sets is the largest partition, and the trivial partition ν consisting of the single element X is the smallest. For a measurable partition ξ there exists a system of *conditional measures* $\mu(\cdot|\xi(x))$ uniquely determined

(7)Borel measures will always be assumed to be regular and complete.

by the following properties:

1) $\mu(\cdot|\xi(x))$ is a probability measure concentrated on $\xi(x)$;

2) if $\varphi \in L_1(\mu)$, then $\int \varphi\, d\mu = \int d\mu(x) \int \varphi(y)\, d\mu(y|\xi(x))$.

1.2. Endomorphisms of a measure space ([41], [19], [46], and [44], Chapters 1–3). Let $f: X \to X$ be a measurable mapping of a Lebesgue space. It transforms the measure μ on X into the measure $f_*\mu$ as follows: if Y is a measurable subset of X, then $(f_*\mu)(Y) = \mu(f^{-1}Y)$. The measure μ is said to be *invariant* if $f_*\mu = \mu$ (in this case one also says that f preserves the measure μ or that f is an endomorphism of the measure space) and *quasi-invariant* if μ and $f_*\mu$ are equivalent (i.e., mutually absolutely continuous: $\mu \sim f_*\mu$).

Observe at once that Lebesgue measure on the sphere is quasi-invariant with respect to a rational endomorphism (and with respect to an arbitrary smooth endomorphism that is a.e. nondegenerate).

A transformation with a quasi-invariant measure is said to be *ergodic* if there is no partition of X into two invariant subsets of positive measure. The theory of measurable partitions permits us to decompose each transformation with quasi-invariant measure into ergodic components and thereby to reduce the investigation of dynamics to the ergodic case. An idea of the dynamics of ergodic transformations with finite invariant measure is obtained from the individual ergodic theorem, which, roughly speaking, asserts that almost every trajectory is uniformly distributed with respect to the measure μ. In this situation a description of the dynamics is possible only in statistical terms. This applies also to the case when μ is quasi-invariant but there exists an invariant measure equivalent to μ. The problem of the existence of such a measure is one of the central problems in ergodic theory.

Besides ergodicity, there is a whole series of stronger stochastic properties, among which exactness is one of the strongest for noninvertible transformations. A transformation with quasi-invariant measure is said to be *exact* if $\bigwedge_{n=0}^{\infty} f^{-n}\varepsilon = \nu$.

The Bernoulli shift σ of the space Σ_l^+ (see Chapter 2, 3.2), endowed with the product measure, is an example of an exact endomorphism. The exactness of this endomorphism is equivalent to the classical 0-1 law for a sequence of independent random variables.

We assume below without specific mention that *the complete inverse image $f^{-1}x$ of almost every point $x \in X$ is at most countable.* Let us consider the partition $\xi = f^{-1}\varepsilon$ into complete inverse images of points.

The *Jacobian* of an endomorphism f with quasi-invariant measure μ is defined to be

$$(J_\mu f)(x) = [\mu(x|\xi(x))\rho(fx)]^{-1},$$

where $\rho(x) = d(f_*\mu)/d\mu$ is the Radon-Nikodým derivative. Intuitively, the Jacobian is equal to the volume expansion coefficient in a neighborhood of x.

A finite or countable measurable partition ξ of a space is called a *one-sided generator* if $\bigvee_{k=0}^{\infty} f^{-k}\xi = \varepsilon$. Not all endomorphisms have a one-sided generator.

Henceforth in this subsection, μ is assumed to be a probability measure and an f-invariant measure.

In classical ergodic theory the isometric operator $U_f: L_2(\mu) \to L_2(\mu)$, $U_f\varphi = \varphi \circ f$, is associated with each endomorphism f of a measure space. The spectral invariants of the endomorphism arise naturally from this. However, for all exact endomorphisms the corresponding operators are similar, and hence the spectral invariants coincide. Therefore, ergodic theory comes to the forefront in the classification of exact endomorphisms (and it is with such endomorphisms that we shall deal below).

The *entropy* of a finite or countable partition $\xi = \{D_i\}_{i=1}^n$ is defined to be $H_\mu(\xi) = -\Sigma\mu(D_i)\ln\mu(D_i)$. If $n < \infty$, then $H_\mu(\xi) \le \ln n$, with equality only if $\mu(D_i) = 1/n$ for all i. In the case of a countable partition it is possible that $H_\mu(\xi) = \infty$.

If ξ and η are two measurable partitions, and the restriction $\xi|\eta(x)$ is at most countable for a.e. x, then the conditional entropy

$$H_\mu(\xi|\eta) = \int H_\mu(\xi|\eta(x))\,d\mu(x)$$

is defined. In particular, we can consider the conditional entropy $H_\mu(\varepsilon|f^{-1}\varepsilon)$, and the following formula holds: $H_\mu(\varepsilon|f^{-1}\varepsilon) = \int \ln(J_\mu f)\,d\mu$.

We shall not give the usual definition of the metric entropy $h_\mu = h_\mu(f)$ of an endomorphism, but only the Rokhlin formula actually to be used.

THEOREM 3.1 [41]. *Assume that the endomorphism f has a one-sided generator with finite entropy. Then*

$$h_\mu(f) = H_\mu(\varepsilon|f^{-1}\varepsilon) = \int \ln(J_\mu f)\,d\mu.$$

The entropy h_μ can thus be interpreted as the mean logarithm of the Jacobian.

In conclusion we describe the construction of the *natural extension* of an endomorphism, which construction assigns an invertible transformation to a noninvertible one. Consider the space $\widehat{X}$ of sequences $\hat{x} = (x_0, x_1, \dots)$ ($x_n \in X$) such that $fx_n = x_{n-1}$. Let $\pi\colon \widehat{X} \to X$ be the natural projection $\hat{x} \mapsto x_0$. We define an invertible transformation $\hat{f}\colon \widehat{X} \to \widehat{X}$ as follows: $\hat{f}\hat{x} = (fx_0, x_0, x_1, \dots)$. The measure μ can be lifted in a unique way to an $\hat{f}$-invariant measure $\hat{\mu}$ on $\widehat{X}$. The transformation of f is ergodic if and only if $\hat{f}$ is such that $h_{\hat{\mu}}(\hat{f}) = h_\mu(f)$.

1.3. Expanding endomorphisms (see [130], [144]). We confine ourselves to the smooth case, though much of what follows is valid in a more general situation. Let $f\colon M \to M$ be a smooth endomorphism of a Riemannian manifold and let $X \subset M$ be an invariant compact set. The endomorphism f is said to be expanding on X if there exist a neighborhood $U \supset X$ and constants $C > 0$ and $\lambda > 1$ such that:

a) $f^{-1}X \cap U = X$;

b) $\|Df^n(x)v\| \ge C\lambda^n\|v\|$ for $x \in X$ and $v \in T_xM$. For example, a rational endomorphism satisfying axiom A is expanding on the Julia set.

We present the following fact for comparison with the results in 6.2 of Chapter 2.

THEOREM 3.2 (see [38], [152]). *An invariant compact set X on which a smooth endomorphism is expanding is structurally stable.*

This means that for each C^1-close endomorphism $g: M \to M$ there exists a g-invariant set X_g on which g is conjugate to $f|X$, with the conjugating homeomorphism $h_g: X \to X_g \subset M$ continuously dependent on g.

1.4. Topological Markov chains and symbolic dynamics (see [6], [130], and [44], Chapter 7). Let A be a matrix of 0's and 1's. Denote by Σ_A^+ the space of one-sided sequences $\{\beta_i\}_{i=0}^{\infty}$ in l symbols with *admissible transitions*: $A_{\beta_i\beta_{i+1}} = 1$. The restriction of the shift $\sigma: \Sigma_l^+ \to \Sigma_l^+$ to Σ_A^+ is called a *(one-sided) topological Markov chain (TMC)*.

A *Markov covering* for the transformation $f: X \to X$ is defined to be a covering of X by closed sets D_i $(1 \le i \le l)$ such that

a) $D_i = \overline{\operatorname{int} D_i}$;
b) $\operatorname{int} D_i \cap \operatorname{int} D_j = \varnothing$, $i \neq j$;
c) f is injective on D_i;
d) fD_i is a union of some of the sets D_j.

To a Markov covering there corresponds a TMC with $l \times l$ matrix A: $A_{ij} = 1$ if $fD_i \supset D_j$. For an admissible sequence $(i_0, \dots, i_{n-1})$ of this TMC we set $D_{i_0} \dots i_{n-1} = \bigcap_{k=0}^{n-1} f^{-k} D_{i_k}$.

A Markov covering is said to be a *topological Markov generator* if $D_{i_0 \cdots i_{n-1}} \to 0$, $n \to \infty$. A transformation f having a topological Markov generator is semiconjugate to the TMC $\sigma: \Sigma_A^+ \to \Sigma_A^+$. The intertwining map $h: \Sigma_A^+ \to X$ assigns to a sequence $i = (i_0, i_1, \dots) \in \Sigma_A^+$ the unique point in the set $\bigcap_{n=1}^{\infty} D_{i_0 \cdots i_{n-1}}$.

THEOREM 3.3 (see [98], [130]). *Let $f: M \to M$ be a smooth endomorphism that is expanding on an invariant compact set $X \subset M$. Then the restriction $f|X$ has a topological Markov generator. The intertwining map $h: \Sigma_A^+ \to X$ is one-to-one to within a set of Baire first category.*

This result enables us to study expanding endomorphisms with the help of symbolic dynamics.

1.5. Pressure, topological entropy, and the variational principle. We mention at once the basic literature on subsections 1.5–1.7: [42], [6], [130], and [44], Chapter 7. The *pressure* of f corresponding to a function φ is denoted by $P_f(\varphi)$. We do not present the usual definition, which is not used below. The variational principle formulated below (Theorem 3.4) can be taken as the definition of the pressure if desired.

Henceforth in this section, all measures are assumed to be Borel probability measures.

In view of known theorems of functional analysis a continuous transformation f of a compact set X always has an invariant measure, and the space M_f of such measures is a compact convex set in the weak topology. The extreme points of this compact are ergodic measures. The set of ergodic measures is denoted by M_f^e.

THEOREM 3.4. *Let $f: X \to X$ be a continuous transformation of a compact set, and let $\varphi \in C(X)$. Then*

$$\sup_{\mu \in M_f} \left(h_\mu(f) + \int \varphi \, d\mu \right) = \sup_{\mu \in M_f^e} \left(h_\mu(f) + \int \varphi \, d\mu \right) = P_f(\varphi).$$

The pressure corresponding to the zero function is called the *topological entropy*: $h(f) = P_f(0)$. In this case the variational principle takes the following form:

$$\sup_{\mu\in M_f} h_\mu(f) = \sup_{\mu\in M_f^e} h_\mu(f) = h(f).$$

The topological entropy is closely connected with homological invariants of the transformation f (see [89]). We present a result of this kind:

THEOREM 3.5 [120]. *Let $f: M \to M$ be a smooth endomorphism of a compact manifold. Then $h(f) \geq \ln|\deg f|$.*

1.6. Gibbs measures. An extremal measure of the variational principle is called a *Gibbs measure.* Gibbs measures need not exist, but if they do exist, then there must be ergodic measures among them. In the case $\varphi = 0$ a Gibbs measure is called a *maximal entropy measure.*

THEOREM 3.6. *Suppose that $f: X \to X$ is an expanding mixing endomorphism of a compact space, and φ is a Hölder function on X. Then f has a unique Gibbs measure μ_φ corresponding to the function φ. Further,* $\operatorname{supp}\mu_\varphi = X$, *and the endomorphism f of the Lebesgue space (X, μ_φ) is exact.*

1.7. Construction of Gibbs measures with the help of the Ruelle operator. With an expanding endomorphism $f: X \to X$ and a function $\varphi \in C(X)$ we associate the *Ruelle operator* $A_\varphi: C(X) \to C(X)$

$$(A_\varphi g)(x) = \sum_{y\in f^{-1}x} e^{\varphi(y)} g(y), \tag{3.1}$$

where $g \in C(X)$. The following result is called Ruelle's variant of the Perron-Frobenius theorem.

THEOREM 3.7. *Assume that the endomorphism f is expanding, and the function φ is Hölder. Let r be the spectral radius of the Ruelle operator. Then r is a simple eigenvalue of the operators A_φ and A_φ^*. Corresponding to it are a positive eigenfunction $h \in C(X)$ and an eigenmeasure ν with* $\operatorname{supp}\nu = X$. *If h is normalized so that $\int h\,d\nu = 1$, then*

$$\frac{1}{r^n} A_\varphi^n g \to \left(\int g\,d\nu\right) h, \qquad n \to \infty$$

for all $g \in C(X)$. The measure ν is the unique quasi-invariant measure such that $J_\nu f = re^{-\varphi}$.

We clarify the nature of this theorem from the point of view of functional analysis in the next subsection.

THEOREM 3.8. *Under the assumptions of the preceding theorem the pressure $P_f(\varphi)$ is equal to $\ln r$, and $\mu = h\nu$ is a Gibbs measure of f corresponding to the function φ.*

The use of the Ruelle operator in constructing and investigating Gibbs measures goes far beyond the framework of the model expanding situation. It is used in an especially interesting way in the theory of one-dimensional endomorphisms, both real (see [65]) and complex (see below). However, there is not yet a sufficiently general description of its domain of applicability.

1.8. The Perron-Frobenius theory for almost periodic operators (see [33]). A bounded operator $A: \mathfrak{B} \to \mathfrak{B}$ in a Banach space is said to be *almost periodic* if the orbit $\{A^n \psi\}_{n=0}^{\infty}$ of an arbitrary vector $\psi \in \mathfrak{B}$ is precompact.

An operator $A: C(X) \to C(X)$ is said to be *nonnegative* if $\psi \geq 0 \Rightarrow A\psi \geq 0$. A nonnegative operator is said to be *primitive* if $\forall \psi \geq 0$, $\psi \neq 0$ $\exists n \in \mathbf{N}$: $A^n \psi > 0$.

THEOREM 3.9. *Let $A: C(X) \to C(X)$ be a primitive almost periodic operator with spectral radius 1. Then there exists a unique nonnegative A-invariant function $h \in C(X)$ and an A^*-invariant measure ν, normalized by the condition $\int h\, d\nu = 1$. Further, $h > 0$ and* $\operatorname{supp} \nu = X$. *For every $\psi \in C(X)$* (8)

$$A^n \psi \to \left(\int \psi\, d\nu \right) h, \qquad n \to \infty.$$

This theorem clarifies the meaning of the various conditions in Ruelle's variant of the Perron-Frobenius theorem. The almost periodicity of the operator is ensured by the fact that f is expanding and φ satisfies a Hölder condition, and its primitivity is ensured by the fact that f is mixing [31].

1.9. Measurable dynamics of expanding endomorphisms. Let M be a compact Riemannian manifold, ν the Riemannian volume on M, and $f: M \to M$ an endomorphism of class C^2 that is expanding on an invariant compact set X.

THEOREM 3.10 (cf. [6]). *Under the above assumptions $\nu(X) = 0$ if X is nowhere dense.*

Assume now that $X = M$, and consider the function $\varphi(x) = -\ln J_\nu f(x)$. Since $\varphi \in C^1$, the results in 1.7–1.8 are applicable: there exists a unique Gibbs measure μ_φ, and it can be constructed with the help of the Ruelle operator A_φ as $h\nu_0$, where h and ν_0 are an eigenfunction and an eigenmeasure of the operators A_φ and A_φ^*, respectively.

It can be shown that the spectral radius of A_φ is equal to 1, and hence $P_f(\varphi) = 0$ (Theorem 3.8). Furthermore, by the change of variable rule, the Riemannian volume is an invariant measure of the operator A_φ, i.e., $\nu_0 = \nu$. The invariant function h of A_φ satisfies the equation

$$h(x) = \sum_{y \in f^{-1}x} \frac{h(y)}{J_\nu f(y)},$$

which means that h is the density of the invariant measure, which is absolutely continuous with respect to the Riemannian volume. Thus, we have

THEOREM 3.11 [99]. *Let $f: M \to M$ be a C^2-smooth expanding endomorphism of a compact manifold M. Then f has a unique invariant measure μ that is equivalent to the Riemannian volume ν. This measure is the Gibbs measure for the function $\varphi = -\ln J_\nu f$. The endomorphism f of the space (M, μ) is exact. The measure μ is the unique invariant measure satisfying the Pesin formula* [39]

$$h_\mu(f) = \int \ln(J_\nu f)\, d\mu.$$

The last assertion follows from the variational principle.

(8)The convergence is understood in the strong topology of the space $C(X)$.

§2. Regular and stochastic dynamics of rational endomorphisms

In this section we investigate the measurable dynamics of rational endomorphisms with respect to Lebesgue measure.

2.1. Regular dynamics: orbits converge to cycles a.e. The following is one of the central problems in the theory of iterates of rational functions.

CONJECTURE. *If* $J(f) \neq \overline{\mathbf{C}}$, *then* $\operatorname{meas} J(f) = 0$.(9)

In view of Theorem 3.10 this conjecture is valid if the endomorphism f satisfies axiom A. Using the Koebe distortion theorem, we prove a more general result. Let $\mathbf{U}_r = \{r: |\overline{z}| < r\}$.

KOEBE DISTORTION THEOREM (see [12]). *Suppose that the function* $\psi: \mathbf{U}_r \to \mathbf{C}$ *is univalent, and let* $q \in (0, 1)$. *Then there exists a constant* $K = K(q)$ *independent of* φ *and* r *such that*

$$\frac{\|D\varphi(x_1)\|}{\|D\varphi(x_2)\|} \leq K$$

for all $x_1, x_2 \in \mathbf{U}_{qr}$.

Let $B(x, \varepsilon) = \{y: \rho(x, y] < \varepsilon\}$ be a disk in the spherical metric, and $r_n(x)$ the spherical radius of the maximal disk about $f^n(x)$ in which the branch of f^{-n} with $f^{-n}(f^n x) = x$ is univalent.

LEMMA 3.1 [25]. *Let* X *be an invariant measurable set contained in* $J(f)$ *with* $\operatorname{meas}(\overline{\mathbf{C}} \setminus X) > 0$. *Then* $r_n(z) \to 0$ *as* $n \to \infty$ *for a.e.* $z \in X$.

PROOF. Let $z \in X$ be such that $\overline{\lim}_{n\to\infty} r_n(z) > 0$. Then $r_{n_k}(z) \geq 2\varepsilon > 0$ for some sequence $\{n_k\}$. Consequently, there exists a neighborhood $D_{\delta,k}$ of the point z such that f^{n_k} maps $D_{\delta,k}$ univalently onto $B(f^{n_k} z, \delta)$, $\delta < 2\varepsilon$. By the Koebe theorem,

$$\frac{\operatorname{meas}(D_{\varepsilon,k} \setminus X)}{\operatorname{meas} D_{\varepsilon,k}} \geq C \frac{\operatorname{meas}(B(f^{n_k} z, \varepsilon) \setminus X)}{\varepsilon^2}, \tag{3.2}$$

where C does not depend on k. It is not hard to show that the expression on the right-hand side of (3.2) is bounded away from zero by a constant independent of k. According to the Koebe theorem, $D_{\varepsilon,k}$ is an oval with bounded distortion, and hence (in view of Theorem 2.4) $\operatorname{diam} D_{\varepsilon,k} \to 0$, $k \to \infty$. Consequently, the lower density of X at z is less than 1. The Lebesgue theorem on points of density concludes the proof. ●

COROLLARY. *Assume that* $J(f) \neq \overline{\mathbf{C}}$. *Then* $\omega(z) \subset \bigcup_c \omega(c)$ *for a.e.* $z \in J(f)$, *where* c *runs through the set of critical points lying on* $J(f)$.

A consequence of this is

THEOREM 3.12 ([25], [76]). *Assume that the orbits of all the critical points converge to cycles, and these cycles are not neutral irrational cycles. Then the following alternative holds*: a) $J(f) = \overline{\mathbf{C}}$; *or* b) $\operatorname{meas} J(f) = 0$.

2.2. Global convergence of the Newton iterative process. Let $P(z) = z^d + a_1 z^{d-1} + \cdots + a_d$ be a complex polynomial with $d > 1$. The Newton iterative

(9)This conjecture has an unsolved analogue in the theory of Kleinian groups: the well-known Ahlfors problem.

process is one of the main numerical methods for finding roots of the polynomial $p(z)$. The sequence of approximations z_n constructed by this method is an orbit of the rational endomorphism $f\colon z \mapsto z - P(z)/P'(z)$. Assume that the roots α_i of P are simple. Then $\deg f = d$, and the α_i are superattracting fixed points of f. Moreover, f has the repelling fixed point ∞. The points α_i and the roots of the polynomial $P''(z)$ are critical points of f.

If the initial approximation z_0 is sufficiently close to α_i, then the Newton process converges to α_i at a superexponential rate. On the other hand, it is at once clear that convergence can fail for some initial approximations: it automatically fails when $z_0 \in J(f)$. But convergence can fail even on the set $F(f)$. The main reason for failure of convergence is the presence of attracting cycles of order > 1 (example: $P(z) = z^3 - z + 1/\sqrt{2}$). Also connected with it are general results of McMullen on the absence of iterative algorithms (see Chapter 2, §6). However, there are some positive results, too.

THEOREM 3.13 (see [29], [134]). *Assume that the roots of the polynomial P are real and simple. Then*: a) *the Newton process $\{z_n\}$ converges to one of the roots for a.e. $z_0 \in \mathbf{C}$*; b) *the Newton process converges to one of the roots for a.e. $z_0 \in \mathbf{R}$ (with respect to linear measure).*

The fact of the matter is that under the conditions of the theorem of orbits of the inflection points of P converge to roots of P, and Theorems 2.19 and 3.10 are applicable.

We mention also Manning's paper [117], in which an algorithm is presented for finding an initial approximation z_0 such that the Newton process converges to one of the roots.

In conclusion we dwell on a simple example.

EXAMPLE 3.1. Consider the Newton process $f\colon z \mapsto \frac{1}{d}[(d-1)z + a/z^{d-1}]$ for finding the roots of the equation $z^d = a$. The only critical point of f different from the root $\sqrt[d]{a}$ is the point $c = 0$. Further, $f\colon c \mapsto \infty \mapsto \infty$, i.e., the endomorphism f is critically finite. By Theorem 3.12, its orbits converge to roots a.e.

Many impressive computer pictures concerned with the Newton iteration process have been made in recent years ([67], [151]).

2.3. The existence of an absolutely continuous invariant measure, and ergodicity.

THEOREM 3.14. *Let f be a critically finite endomorphism without superattracting cycles. Then f has a unique invariant measure ν equivalent to Lebesgue measure. The endomorphism f of the space $(\overline{\mathbf{C}}, \nu)$ is exact, and the Pesin formula holds*:

$$h_\nu(f) = 2\int \ln \|D_f\|\, d\nu > 0.$$

This theorem can be proved with the use of the natural metric of the orbifold $\mathscr{O}_f$ (Chapter 2, §5) and the Ruelle operator in the same way as in the case of expanding endomorphisms (subsection 1.9). The set Λ of endomorphisms satisfying the conditions of Theorem 3.14 has zero measure in the space $\mathfrak{R}_d$ of all rational functions of degree d. However, as shown by the following result of Mary Rees, stochasticity holds for an essentially larger class of endomorphisms.

THEOREM 3.15 [124]. *The set of ergodic endomorphisms (with respect to Lebesgue measure) has positive measure in the space $\mathfrak{R}_d$.*

Note that the ergodic endomorphisms constructed in this theorem are in the closure of the set Λ.

THEOREM 3.16 (Sullivan [139]). *$J(f)$ does not contain wandering sets X of positive measure on which all iterates $f^n|X$ are injective.*

This property (proved with the help of quasiconformal deformation techniques) can be regarded as weak conservativity on the Julia set. It is unknown whether $f|J$ is conservative in the strong sense (i.e., whether the recurrence theorem holds).

2.4. The characteristic exponent and local instability of a manifold. The local manifold technique is a powerful apparatus for investigating smooth dynamical systems. It was developed by Pesin [39] for diffeomorphisms in a very general context [39]. The difficulties connected with noninvertibility can be overcome by passing to the natural extension. We confine ourselves to the case of a rational endomorphism.

Let μ be an invariant measure of a rational endomorphism f. By the ergodic theorem, the limit

$$\chi_\mu(x) = \lim_{n\to\infty} \frac{1}{n} \ln \|Df^n(x)\| \in [-\infty, \infty)$$

exists for μ-a.e. $x \in \overline{\mathbf{C}}$, and $\int \chi_\mu \, d\mu = \int \ln \|Df\| \, d\mu$. The quantity $\chi_\mu(x)$ is called the *characteristic exponent.* It characterizes the exponential instability a.e. of trajectories. If μ is ergodic, then the characteristic exponent does not depend on x a.e., and is equal to $\chi_\mu = \int \ln \|Df\| \, d\mu$. The following result is a special case of the Margulis-Ruelle inequality (see [131]):

$$h_\mu(f) \leq 2 \max \left(\int \ln \|Df\| \, d\mu, 0 \right). \tag{3.3}$$

it shows that the positivity of the entropy has to do with the exponential instability a.e. of the trajectories.

The *diameter of a partition* η is defined to be the supremum of the diameters of its elements. A partition η is said to be *μ-open* if it consists mod 0 of open sets, and *injective* if $f|D_i$ is injective for all $D_i \in \eta$.

Let $\hat{f} \colon \widehat{\mathbf{C}} \to \widehat{\mathbf{C}}$ be the natural extension of the endomorphism f, $\pi \colon \widehat{\mathbf{C}} \to \mathbf{C}$ the natural projection, and $\hat{\mu}$ the lifting of the measure μ.

THEOREM 3.17 ON UNSTABLE MANIFOLDS (Ledrappier [100], [101]). *Suppose that $\chi_\mu(x) > 0$ for μ-a.e. x. Then there exists a countable μ-open injective partition η of the sphere $\overline{\mathbf{C}}$ having arbitrarily small diameter and finite entropy such that:*

a) *η is a one-sided generator of f;*

b) *the partition $\xi = \bigvee_{k=0}^{\infty} \hat{f}^k(\pi^{-1}\eta)$ (mod 0) consists of sets whose projections on $\overline{\mathbf{C}}$ are open;*

c) *the uniform estimate*

$$C_1(\hat{x}) \leq \frac{\|Df^n(x_n)\|}{\|Df^n(y_n)\|} \leq C_2(\hat{x})$$

of distortion holds on $\xi(\hat{x})$, where $\hat{x} = \{x_n\}_{n=0}^{\infty}$ and $\hat{y} = \{y_n\}_{n=0}^{\infty} \in \xi(\hat{x})$;

d) $\lim_{n\to\infty} \frac{1}{n} \ln \rho(x_n, y_n) = -\chi_\mu(x_0)$ *for $\hat{y} \in \xi(\hat{x})$.*

Note that $\hat{y} \in \xi(\hat{x})$ means that for all $n \in \mathbf{N}$ the points x_n and y_n are contained in a single element of the partition η. The sets $\xi(\hat{x})$ are called *local unstable manifolds.* Theorem 3.17 enables us to regard an endomorphism with positive characteristic exponent as nonuniformly expanding.

COROLLARY. *If* $\chi_\mu(x) > 0$ *a.e., then* $\overline{\lim}_{n\to\infty} r_n(x) > 0$ *for a.e.* x ($r_n(x)$ *is defined in* 2.1).

PROOF. Consider the set $Y_\alpha \subset \widehat{\mathbf{C}}$ consisting of those $\hat{x}$ such that $\pi\xi(\hat{x})$ contains the disk $B(x_0, \alpha)$. By the Poincaré recurrence theorem, there exists for a.e. $\hat{x} \in Y_\alpha$ a sequence $n_k \to \infty$ such that $\hat{f}^{n_k}\hat{x} \in Y_\alpha$. This implies that $r_{n_k}(x_0) \geq \alpha$. Since $\hat{\mu}(\bigcup_{\alpha>0} Y_\alpha) = 1$, the required inequality has been verified. ●

2.5. Stochastic properties of endomorphisms having an a.c.i. measure. (10) Ledrappier showed that if a rational endomorphism has an a.c.i. measure with positive characteristic exponent, then it has all the properties inherent in Gibbs distributions (see 1.6).

THEOREM 3.18 ([100], [101]). *Let* μ *be an* f*-invariant measure with positive characteristic exponent:* $\chi_\mu(x) > 0$ *for* μ*-a.e.* x. *Then the following properties are equivalent:*

a) μ *is absolutely continuous with respect to Lebesgue measure.*

b) *The Pesin formula*

$$h_\mu(f) = 2\int \chi_\mu \, d\mu = 2\int \ln \|Df\| \, d\mu$$

is valid. Further, $J(f) = \operatorname{supp}\mu = \overline{\mathbf{C}}$, *and the endomorphism* f *of the space* $(\overline{\mathbf{C}}, \mu)$ *is exact. The measure with the indicated properties is unique.*

SOME IDEAS IN THE PROOF. If a) is satisfied, then

$$J_\mu f(z) = \frac{\psi(fz)}{\psi(z)}\|Df(z)\|^2, \tag{3.4}$$

where ψ is the density of the measure μ. By Theorem 3.17, f has a one-sided generator. Therefore, Theorem 3.1 is applicable, and together with (3.4) it implies b). In proving the reverse implication Ledrappier explicitly writes out the density of μ. The property $\operatorname{supp}\mu = \overline{\mathbf{C}}$ and the ergodicity follow from Lemma 3.1 and the corollary to Theorem 3.17 (cf. [25]). The ergodicity implies that μ is unique.

§3. Hausdorff dimension

3.1. Definition. Let X be a metric space with metric ρ, and let $Y \subset X$. An ε-covering of Y is defined to be a covering by balls $B(x_i, r_i)$ of radius less than ε. For $\delta > 0$ let

$$l_\delta(Y, \varepsilon) = \inf \sum_f r_i^\delta,$$

where the infimum is over all ε-coverings of Y. Then $l_\delta(\cdot) = \lim_{\varepsilon\to 0} l_\delta(\cdot, \varepsilon)$ is a Borel measure on X. It is called the *Hausdorff measure* corresponding to the exponent δ.

(10)That is, an absolutely continuous (with respect to Lebesgue measure) invariant measure.

To define the Hausdorff dimension we study the dependence of the Hausdorff measure on δ. It turns out that for every set $Y \subset X$ there exists a value $\delta(Y)$ such that $l_\delta(Y) = 0$ for $\delta > \delta(Y)$ and $l_\delta(Y) = \infty$ for $\delta < \delta(Y)$. This value of the exponent is called the *Hausdorff* (or *fractal*) *dimension* of the set Y: $\dim Y = \delta(Y)$. The reader can acquaint himself with details of the construction and elementary properties of the dimension in the book [5].

The (Borel) *dimension of a measure* μ on a set X is defined to be

$$\dim \mu = \inf_{Y\,:\,\mu(X\setminus Y)=0} \dim Y.$$

3.2. Dimension, characteristic exponent, and entropy. The quantities in the heading are closely connected in a fairly general situation, but this connection usually bears the character of estimates (see [44], Chapter 7). For conformal systems they pass into equalities.

THEOREM 3.19 [101]. *Let f be a rational endomorphism, and μ an invariant ergodic measure. Then*([11])

$$h_\mu(f) = \max(\chi_\mu, 0) \dim \mu.$$

Further, for a.e. x

$$\dim \mu = \lim_{\varepsilon \to 0} \frac{\ln \mu(B(x, \varepsilon))}{\ln \varepsilon}.$$

COROLLARY 1. *An invariant measure with positive entropy has positive dimension.*

COROLLARY 2. *The Julia set always has positive dimension*: $\dim J(f) > 0$.

PROOF. By Theorem 5.5 (Misiurewicz-Przytycki), if $\deg f > 1$, then $h(f) > 0$. Consequently (the variational principle), f has an invariant ergodic measure μ of positive entropy. Since $\chi_\mu > 0$, it follows that $\operatorname{supp} \mu \subset J(f)$. Thus, $\dim J(f) \geq \dim \mu > 0$. ●

3.3. The Bowen formula and the conformal measures. If a rational endomorphism satisfies axiom A, then the restriction $f|J$ is an expanding endomorphism, and the general theory in §1 is applicable to it. The pressure of the endomorphism $f|J$ corresponding to the function φ will be denoted by $P(\varphi)$.

THEOREM 3.20 ([62], [132]). *Suppose that the rational endomorphism f satisfies axiom A. Then the equation*

$$P(-\delta \ln \|Df\|) = 0 \tag{3.5}$$

has a unique root $\delta = \dim J$ equal to the Hausdorff dimension of the Julia set.

The formula (3.5) bears the name of Bowen, who discovered an analogue of it for quasi-Fuchsian groups. Further results were obtained by Sullivan [138]. In the formulations below we set $\delta = \dim J$.

THEOREM 3.21. *If the endomorphism f satisfies axiom A, then $0 < l_\delta(J) < \infty$ and $0 < \dim J < 2$. The measure l_δ is quasi-invariant and exact with respect to f.*

A measure m is said to be *conformal* (with exponent δ) for the rational endomorphism f if $J_m f(z) = \|Df(z)\|^\delta$ a.e.

([11])Cf. the Margulis-Ruelle inequality (3.3).