THEOREM 3.22. *Let $f$ satisfy axiom $A$. Then the normalized Hausdorff measure $l_\delta$ is the unique conformal probability measure with exponent $\delta = \dim J$. The Gibbs measure $\mu_\delta$ corresponding to the function $\varphi_\delta = -\delta \ln \|Df\|$ is equivalent to the measure $l_\delta$.*

THEOREM 3.23. *Let $f$ satisfy axiom $A$. Then the Gibbs measure $\mu_\delta$ is the unique invariant ergodic measure with dimension equal to $\delta = \dim J$.*

PROOFS OF THE ABOVE THEOREMS. We assume without proof the existence of a unique root $\delta$ of equation (3.5). Consider the Ruelle operator (3.1) corresponding to the function $\varphi_\delta = -\delta \ln \|Df\|$:

$$(A\psi)(z) = \sum_{\zeta \in f^{-1}z} \frac{\psi(\zeta)}{\|Df(\zeta)\|^\delta}.$$

The spectral radius of $A$ is equal to $\exp P(\varphi_\delta) = 1$. By Ruelle's variant of the Perron-Frobenius theorem, there exists a unique $A^*$-invariant probability measure $m$. But $A^*$-invariance is equivalent to conformality. Thus, $m$ is the unique conformal probability measure.

We now consider a Markov generator $\{D_i\}$ on the Julia set (see 1.4). By the Koebe distortion theorem, the sets $D_{i_0 \cdots i_{n-1}}$ are ovals with uniformly bounded distortion, and

$$m(D_{i_0\cdots i_{n-1}}) \asymp \|Df^n(x_{i_0\cdots i_{n-1}})\|^{-\delta} \asymp (\operatorname{diam} D_{i_0\cdots i_{n-1}})^\delta,$$

where $x_{i_0\cdots i_{n-1}}$ is an arbitrary point in $D_{i_0\cdots i_{n-1}}$. This implies that there exist constants $C_1, C_2 > 0$ such that $C_1 m(X) \le l_\delta(X) \le C_2 m(X)$ for every measurable set. In particular, $C_1 \le l_\delta(J) \le C_2$, and hence $\delta = \dim J$, which proves the Bowen formula.

Further, the measure $l_\delta$ is obviously conformal, and since the conformal probability measure is unique, it follows that $l_\delta / l_\delta(J) = m$. The equivalence of the Hausdorff measure $l_\delta$ and the Gibbs measure $\mu_\delta$ follows from Theorem 3.8. The measure $l_\delta$ is exact, because $\mu_\delta$ is. Since $l_\delta(J) > 0$ while $l_2(J) = 0$ (Theorem 3.10), it follows that $\delta < 2$. The fact that $\delta > 0$ was established in the preceding subsection.

It remains to prove that $\dim \nu < \delta$ for every ergodic measure $\nu \ne \mu_\delta$. To do this we apply the variational principle at the function $\varphi_\delta$:

$$h_\nu - \delta \chi_\nu = h_\nu + \int \varphi_\delta \, d\nu < P(\varphi_\delta) = 0.$$

By Theorem 3.19, $\dim \nu = h_\nu / \chi_\nu < \delta$. ●

We remark that some of the results proved here can be extended to arbitrary rational endomorphisms [138].

**3.4. The fractal property of quasicircles.** We consider a rational endomorphism satisfying axiom $A$ with two components of the Fatou set. In this case the Julia set $J$ is a Jordan curve (Chapter 2, §4). The next result shows that, as a rule, this cure is *fractal*, i.e., its Hausdorff dimension exceeds its topological dimension (see [112]).

THEOREM 3.24 [138]. *The following alternative holds under the assumptions made above*: a) *$J$ is a circle*; b) $1 < \dim J < 2$.

PROOF. If a) does not hold, then the curve $J$ is not rectifiable (see Chapter 2, §4.3). Consequently, $l_1(J) = \infty$. On the other hand, $l_\delta(J) < \infty$ by Theorem 3.21, where $\delta = \dim J$. Consequently, $\delta > 1$. ●

Theorem 3.24 is an analogue of an earlier theorem of Bowen [62] relating to limit sets of quasi-Fuchsian groups. In [123] an analogous result was obtained for the class of Jordan curves in a neighborhood of which there is piecewise-conformal dynamics; this class includes both Julia curves and limit sets of quasi-Fuchsian groups.

**3.5. Analytic dependence of dimension on parameters.**

THEOREM 3.25 (Ruelle [132]). *Consider an A-domain $W$ of the space $\mathfrak{R}_d$ of rational endomorphisms of degree $d$. Then* $\dim J(f)$ *is a real-analytic function on $W$.*

The proof is based on Bowen's formula. Ruelle showed that $P(-\delta \ln \|Df\|)$ depends analytically on $(\delta, f)$, and the implicit function theorem can be applied to equation (3.5).

The first terms of the decomposition of the function $\delta(\varepsilon) = \dim J(z^2 + \varepsilon)$ in a neighborhood of zero are given by (see [132]) $\delta(\varepsilon) = 1 + |\varepsilon|^2/4\ln 2 + o(|\varepsilon|^2)$.

## §4. The maximal entropy measure

**4.1. Convergence of powers of the averaging operator.** Consider the averaging operator

$$(A\varphi)(z) = \frac{1}{d} \sum_{\zeta \in f^{-1}z} \varphi(\zeta), \tag{3.6}$$

where the points in the inverse image $f^{-1}z$ are regarded with multiplicity taken into account. In view of the continuous dependence of the inverse image on the point this operator acts in the space $C(\overline{\mathbf{C}})$ of continuous functions. Further, $A$ is *stochastic*, i.e., $A \geq 0$ and $A\mathbf{1} = \mathbf{1}$, where $\mathbf{1}$ is the function identically equal to 1. The operator $A$ can be regarded as the Ruelle operator (3.1) corresponding to the function $\varphi = -\ln d$.

Let $K_\varepsilon$ be the complement of the $\varepsilon$-neighborhood of the exceptional points of the endomorphism $f$ (see Chapter 2, §1.3). Since the compact set $K_\varepsilon$ is $f^{-1}$-invariant for sufficiently small $\varepsilon$, (3.6) unambiguously determines the factor of $A$ on $C(K_\varepsilon)$, which we also denote by $A$.

LEMMA 3.2 ([81], [63]). *Let $f_i^{-n}$ be branches of the inverse functions and assume that they are single-valued in the neighborhood $V$. Then the family $\{f_i^{-n}\}_{n,i}$ is normal.*

THEOREM 3.26 ([24], [102]). *There exists a unique $A^*$-invariant probability measure $\mu$. Further,* $\operatorname{supp}\mu = J(f)$, *and in the space $C(K_\varepsilon)$*

$$A^n\varphi \to \left(\int \varphi\, d\mu\right)\mathbf{1}, \qquad n \to \infty.$$

The proof is based on the Perron-Frobenius theory for almost periodic operators (see 1.8). Lemma 3.2, which is a shadow of axiom $A$, is basic for almost periodicity of the averaging operator on $C(K_\varepsilon)$. If $J \neq \overline{\mathbf{C}}$, then $A$ is not primitive on $C(K_\varepsilon)$ (it is primitive on $C(J)$), but a weaker property suffices for convergence of the powers: $\forall\varphi \in C(K_\varepsilon)$, $\varphi \geq 0$, $\varphi | J \not\equiv 0$ $\exists N$ such that

$(A^n\varphi)(x) > 0$ for $n \geq N$ and $x \in K_\varepsilon$. This property follows at once from Theorem 2.4.

Up to the end of §4 the letter $\mu$ will denote the measure constructed in Theorem 3.26.

**4.2. The distribution of the inverse images and the periodic points.** We consider the discrete probability measures

$$\mu_{n,z} = \frac{1}{d^n} \sum_{\zeta \in f^{-n} z} \delta_\zeta, \qquad \nu_n = \frac{1}{d^n + 1} \sum_{f^n \zeta = \zeta} \delta_\zeta,$$

where $\delta_\zeta$ is the Dirac measure at the point $\zeta$.

The following result was obtained for polynomials by Brolin [63], and for rational endomorphisms in [24] and [102], then somewhat later by three authors in [85].

THEOREM 3.27. *For an arbitrary nonexceptional point* $z \in \overline{\mathbf{C}}$ *the convergence* $\mu_{n,z} \to \mu$, $n \to \infty$, *holds in the weak topology of the space of measures.*

The proof follows at once from Theorem 3.26:

$$\int \varphi \, d\mu_{n,z} = (A^n \varphi)(z) \to \int \varphi \, d\mu, \qquad n \to \infty.$$

THEOREM 3.28 ([24], [102]). *The asymptotic distribution of the periodic points with respect to the measure* $\mu$ *is uniform. More precisely,* $\nu_n \to \mu$, $n \to \infty$, *in the weak topology.*

This result can be derived from Theorem 3.27 by approximating the inverse images by periodic points.

**4.3. Existence and uniqueness of the maximal entropy measure.**

THEOREM 3.29 ([24], [102]; see also [113]). *The measure* $\mu$ *is the unique maximal entropy measure.*

PROOF. It is easy to verify that $\mu$ is $f$-invariant. The $A_\varepsilon$-invariance of this measure means that the conditional measures $\mu(x|f^{-1}\varepsilon)$ are equal to $1/d$ a.e. Consequently, $J_\mu f = d$ a.e. By the theorem on unstable manifolds, $f$ has a one-sided generator, and hence Theorem 3.1 is applicable: $h_\mu(f) = \int \ln(J_\mu f)\, d\mu = \ln d$.

Since any other $f$-invariant measure $\nu$ is not $A^*$-invariant, it follows that $H_\nu(x|f^{-1}\varepsilon) < \ln d$ on a set of positive $\nu$-measure. If $h_\nu(f) > 0$, then we again use Theorem 3.1:

$$h_\nu(f) = H_\nu(\varepsilon|f^{-1}\varepsilon) = \int H_\nu(x|f^{-1}\varepsilon)\, d\nu < \ln d. \quad \bullet$$

COROLLARY ([23], [88], [102]). *The topological entropy of a rational endomorphism of degree* $d$ *is equal to* $\ln d$.

**4.4. The Bernoulli property.** It was shown in [24], [102], and [85] that the endomorphism $f$ is *exact* with respect to the maximal entropy measure. It is very likely isomorphic to the one-sided Bernoulli shift with equally probable outcomes. In [114] this conjecture was proved for some power $f^n$. Moreover, it is valid if $f$ satisfies axiom $A$. For polynomials this follows from the symbolic dynamics constructed by Jakobson and Guckenheimer:

THEOREM 3.30 ([48], [49], [90]). *Let $f$ be a polynomial satisfying axiom $A$. Then there exists a continuous mapping $\pi : \Sigma_d^+ \to J$ that semiconjugates $\sigma$ and $f|J$ and is one-to-one modulo countable sets.*

## §5. Harmonic measure

**5.1. The dimension of harmonic measure.** Let $D$ be a domain on the sphere $\overline{\mathbf{C}}$. If the boundary $\partial D$ has positive capacity (see [7]), then the Dirichlet problem is uniquely solvable in $D$, and hence there exists a unique measure $\omega_D(\cdot, a)$ on $\partial D$ such that

$$h(a) = \int h(z)\, d\omega_D(z, a)$$

for each harmonic function $h : D \to \mathbf{R}$ that is continuous on $\overline{D}$. This measure is called *harmonic* measure. Variation of the point $a$ leads to replacement of $\omega_D$ by an equivalent measure, and thus we do not always indicate the dependence of $\omega_D$ on $a$.

Harmonic measure exists on the boundary of an arbitrary simply connected domain other than $\overline{\mathbf{C}}$ and $\mathbf{C}$. In this case it can be defined as the image of Lebesgue measure on $\mathbf{T}$ under a conformal mapping $\varphi : \mathbf{U} \to D$ (the definition is unambiguous, since $\varphi$ has radial limits for a.e. $z \in \mathbf{T}$). This definition shows the importance of harmonic measure for the theory of univalent functions. In particular, computation of the Hausdorff dimension of harmonic measure is directly connected with the study of the distortion under a conformal mapping. Important results in this direction have been obtained recently by Makarov.

THEOREM 3.31 [34]. *If $D$ is a simply connected domain other than $\overline{\mathbf{C}}$ and $\mathbf{C}$, then* $\dim \omega_D = 1$. *The harmonic measure $\omega_D$ is absolutely continuous with respect to the Hausdorff measure $l_1$ if and only if the derivative of a conformal mapping $\varphi : \mathbf{U} \to D$ has radial limits for a.e. $z \in \mathbf{T}$.*

**5.2. Harmonic measure on the boundary of an invariant domain.** Harmonic measure arises in the theory of iterates in connection with the following fact.

LEMMA 3.3. *Let $D$ be a domain whose boundary has positive capacity, and $f : \overline{D} \to \overline{D}$ a continuous transformation that is analytic in $D$ such that $f(\partial D) \subset \partial D$. Assume that $f$ has an attracting fixed point $a$ in $D$. Then the harmonic measure $\omega_D = \omega_D(\cdot, a)$ is $f$-invariant.*

PROOF. Let $h$ be a harmonic function on $D$ that is continuous on $\overline{D}$. Then $\int h\, d\omega = h(a) = h(fa) = \int h \circ f\, d\omega$. Since $h|\partial D$ can be an arbitrary continuous function, the measure $\omega$ is $f$-invariant. ●

EXAMPLE 3.3. The harmonic measure $\omega_{\mathbf{U}}(\cdot, 0)$ coincides with Lebesgue measure $\lambda$ on the circle $\mathbf{T}$. Consider a Blaschke product for which 0 is a fixed point. Then $f$ preserves the measure $\lambda$. Since $f|\mathbf{T}$ is an expanding endomorphism, $\lambda$ is its Gibbs measure corresponding to the function $-\ln|f'(z)|$ (see 1.9). By Theorem 3.11, the endomorphism $f$ of the measure space $(\mathbf{T}, \lambda)$ is exact, and the entropy satisfies the formula $h_\lambda(f) = \int \ln|f'|\, d\lambda$.

We mention some additional papers dealing with the dynamics of Blaschke products: [118] and [133].

EXAMPLE 3.4. Let us apply Lemma 3.3 to a polynomial $f$ on the domain $D = D(\infty)$ of attraction of $\infty$. We get that the harmonic measure $\omega = \omega_D(\cdot, \infty)$, which is concentrated on the Julia set $J$, is $f$-invariant. This situation was investigated in 1965 by Brolin:

THEOREM 3.32 [63]. *Let $f$ be a polynomial. Then the inverse images $f^{-n}z$ of every point $z \in \mathbf{C}$ with perhaps one exception have an asymptotically uniform distribution with respect to the harmonic measure $\omega$ on $J(f)$.*

Comparing Theorem 3.32 and the results in §4, we get the

COROLLARY. *For a polynomial $f$ the maximal entropy measure coincides with the harmonic measure $\omega$ on $J(f)$.*

**5.3. Harmonic measure of fractal curves.** Consider the family $f_\varepsilon\colon z \mapsto z^2 + \varepsilon$. As we know (§3.4), for small $|\varepsilon|$ the Julia set $J = J(f_\varepsilon)$ is a fractal curve. Since the dimension of the harmonic measure $\omega$ on $J$ is equal to 1 (Theorem 3.31),([12]) $\omega$ lies on a thin part of the curve $J$: $\omega$ is automatically singular with respect to the Hausdorff measure $l_\delta$.

It turns out that $\omega$ is actually singular even with respect to the measure $l_1$ (cf. Theorem 3.31). This fact can be extended to many fractal self-similar curves of different origin: the Koch curve ("snowflake"), the Dekking curves [68], and limit sets of quasi-Fuchsian groups. Analogous assertions are valid for fractal sets that are not curves. The dynamical approach using the theory of Gibbs measures, Bowen's formula (3.5), and Theorem 3.19 are crucial to this circle of questions. The reader interested in more details about these problems is referred to [64], [105], [106], [123], and [155].

**5.4. Comparison of different measures.** We have considered three natural measures: the conformal measure, the maximal entropy measure, and the harmonic measure. Some connections between them were mentioned in 5.2–5.3 (see also [103]). Anna Zdunik recently solved the problem of comparing the maximal entropy measure $\mu$ with the conformal measure. Let $\alpha = \dim\mu$ and $\delta = \dim J$.

THEOREM 3.33 [146]. *Suppose that the rational endomorphism $f$ is not critically finite with parabolic orbifold. Then*:

a) $\alpha < \delta$;

b) *the maximal entropy measure $\mu$ is mutually singular with the Hausdorff measure $l_\alpha$, and hence with the measure $l_\delta$ (which coincides with the conformal measure on the Julia set in the case of axiom $A$).*

In particular, we consider a polynomial $f$ with connected Julia set. Then $\dim\mu = 1$ (Theorem 3.31 and the corollary to Theorem 3.33). Consequently, if $f$ is not conformally conjugate to $z^d$ or to a Tchebycheff polynomial, then $\dim J > 1$. This shows that the Julia set is fractal. We mention that the term "fractal" was used universally with regard to the Julia set, but this becomes justified only after Zdunik's theorem.

In conclusion we direct the attention of the reader to the fact that the rational functions with parabolic orbifolds (in particular, the polynomials $z^d$ and the Tchebycheff polynomials) arise as remarkable exceptions in very diverse problems of holomorphic dynamics.

## CHAPTER 4
## ITERATES OF ENTIRE FUNCTIONS

We have described in detail the dynamics of analytic endomorphisms of hyperbolic Riemann surfaces and just one elliptic surface—the sphere $\overline{\mathbf{C}}$. Left unstudied are the endomorphisms of the three parabolic Riemann surfaces: $\mathbf{C}$,

([12])See also [116] and [122].

$\mathbf{C}^*$, and $\mathbf{T}^2$. In the case of the torus $\mathbf{T}^2$ an arbitrary analytic endomorphism $f$ is generated by an affine transformation of the plane $\mathbf{C}$, $z \mapsto az+b$. If $\deg f = 1$, then either $f$ has finite order, or the dynamics bear a quasiperiodic character. If $\deg f > 1$, then $f$ belongs to the thoroughly studied class of expanding endomorphisms (see Chapter 3). By passing to a covering endomorphism the case $\mathbf{C}^*$ can to a considerable degree be reduced to the case $\mathbf{C}$ (see [35], [150]), which is the main object of this chapter. The analytic endomorphisms of the plane $\mathbf{C}$ are the entire functions. The investigation of the iterates of entire functions was begun by Fatou in 1926 [84], and then continued mainly by Baker ([53]–[58]). Interest in this circle of problems, as well as in conformal dynamics as a whole, has grown significantly in recent years ([59], [14]–[17], [69]–[72], [86], [91]–[92], [108]).

Let us dwell briefly on the contents of this chapter. In §§1 and 2 we develop the general theory (without special restrictions on the class of functions). Here facts and methods specific to the transcendental case are emphasized. Two theorems of Baker (4.1 and 4.3) are the central results of these sections. In §3 we give some examples of entire functions with "pathological" properties of their dynamics (in comparison to the rational case). §4 is devoted to an important class $S$ of entire functions that are free of the basic pathologies. For the functions of this class the authors have given a complete description of the dynamics of the Fatou set. In the concluding section we describe the dynamical properties of exponential transformations.

## §1. Periodic points

The Fatou set $F(f) \subset \mathbf{C}$ and the Julia set $J(f) \subset \mathbf{C}$ for a transcendental entire function are defined in exactly the same way as for a rational function. Both sets are completely invariant and do not change when $f$ is replaced by $f^n$. Fatou showed [84] that the Julia set $J(f)$ of an entire function is perfect (the proof is more complicated than in the rational case). The set $J(f)$ is nowhere dense or coincides with $\mathbf{C}$. However, in contrast to the polynomial case, the Julia set of a transcendental entire function is always unbounded. In general, infinity has a very different influence on the dynamics in the transcendental case than in the polynomial case.

The concept of an exceptional point $a \in \mathbf{C}$ (see Chapter 12, §1.3) for an entire function is equivalent to the situation when $f^{-1}a = \varnothing$ or $f^{-1}a = \{a\}$. An entire function can have at most one exceptional point. In this case $f$ can be reduced by an affine conjugacy to the form $z \mapsto z^n \exp g(z)$, where $n \in \mathbf{N}$ and $g$ is an entire function. However, the exceptional point is not necessarily fixed, and can lie in the Julia set. For example, if $f\colon z \mapsto e^z$, then $f^n 0 \to \infty$ and $0 \in J(f) = \mathbf{C}$ (see §5). For the function $f\colon z \mapsto 2ze^z$ the exceptional point $0$ is a fixed repelling point.

The first serious difficulties in carrying the rational theory over to the transcendental case arise in proving Theorem 2.3. A repetition of the argument shows only that the points of the Julia set can be approximated by periodic points, but this in no way implies that the latter are repelling (or even that they lie in $J(f)$). This important fact was established by Baker, using the following very deep result [47].

AHLFORS' THREE DOMAINS THEOREM. *Let $E_1$, $E_2$, and $E_3$ be bounded Jordan domains with disjoint closures, and let $r > 0$. Then there exists a constant $C$ with the following property. For every function $f$ holomorphic in the disk $\mathbf{U}_r$*

*and such that* $|f'(0)|/(1+|f(0)|^2) > C$ *there exists a domain* $D$ *contained in* $\mathbf{U}_r$ *and with closure mapped univalently by* $f$ *onto one of the domains* $E_i$.

THEOREM 4.1 (Baker [55]). *The repelling cycles of an entire function are dense in the Julia set.*

PROOF. Let $V$ be an open set with $V \cap J \neq \varnothing$. Since $J$ is perfect, there exist three distinct points $z_j \in J \cap V$. Take disjoint disks $E_j = B(z_j, 2r) \subset V$. The spherical derivatives of the iterates $f^n$ are unbounded in a neighborhood of the points $z_j$ as $n \to \infty$. Therefore, there are points $z'_j \in B(z_j, r)$ and numbers $m_j$ such that $(|(f^{m_j})'|/(1+|f^{m_j}|^2))(z'_j) > C$, where $C$ is the constant in the three domains theorem. According to this theorem, there exist domains $D_j$ such that $\overline{D}_j \subset B(z'_j, r)$, and each $D_j$ is mapped univalently by $f^{m_j}$ onto one of the disks $E_k$. Then there exist $m$ and $k$ such that $f^m$ maps some domain $D \subset D_k$ univalently onto $E_k$. The inverse mapping $E_k \to D$ has an attracting fixed point, since $\overline{D} \subset E_k$. Consequently, $f^m$ has a repelling fixed point in $D \subset V$, which is what was required. ●

Let $I(f) = (z\colon f^n z \to \infty\}$. Using the Wiman-Valiron method (see [8], Chapter IX), Eremenko (Banach Center Publ., Warsaw, 1989) proved the following result.

THEOREM 4.2. $I(f) \neq \varnothing$ *and* $J(f) = \partial I(f)$ *for every nonlinear entire function* $f$. $J(f) \cap I(f) \neq \varnothing$ *for transcendental functions* $f$.

## §2. The components of the Fatou set

**2.1. Simple connectedness of unbounded and nonwandering components.** As we know, a polynomial $f$ has a single unbounded component $D(\infty)$ of the set $F(f)$. If $f$ is regarded as a transformation $\mathbf{C} \to \mathbf{C}$, then this component is multiply connected. On the other hand, all the bounded components of $F(f)$ are simply connected. If $f$ is a transcendental entire function, then the situation is completely different. The set $F(f)$ can have infinitely many unbounded components, or it can have none at all. If the number $n$ of unbounded components is finite and nonzero, then $n = 1$. It is likely that in this case the Fatou set coincides with the unique unbounded component. An unexpected fact is the simple connectedness of the unbounded components. On the other hand, the bounded components can be multiply connected. Such components are necessarily wandering.

The main difference between transcendental entire functions and rational functions is that the former are not in general branched covers $\mathbf{C} \to \mathbf{C}$. If $D$ is a component of the set $F(f)$ and $D_1$ the component containing $fD$, then the mapping $f\colon D \to D_1$ is also not necessarily a branched cover. However, the mapping $f|D$ is a branched cover (by Lemma 1.1) in the case when the component $D$ is bounded. In particular, $fD = D_1$.

Denote the index of a curve $\gamma \subset \mathbf{C}$ with respect to a point $a$ and by $\operatorname{ind}_a \gamma$.

LEMMA 4.1. *Let* $D$ *be a multiply connected component of the Fatou set* $F(f)$, *and* $\gamma$ *a Jordan curve that is not contractible in* $D$. *Then*:

a) $f^n \to \infty$ *uniformly on compact subsets of* $D$;

b) $\operatorname{ind}_0(f^n\gamma) > 0$ *for all sufficiently large* $n$.

PROOF. a) Assume that some subsequence $\{f^{n_k}\}$ is uniformly bounded on compact subsets of $D$. Then $|f^{n_k}| \leq M$ on $\gamma$, and hence $|(f^{n_k})'| \leq C$ inside

$\gamma$. But inside $\gamma$ there are points of $J(f)$, and we obtain a contradiction with Theorem 4.1. b) Assume that $\operatorname{ind}_0(f^{n_k}\gamma) = 0$ for some subsequence $\{n_k\}$. Then $f^{n_k}$ does not have zeros inside $\gamma$ (the argument principle). By the minimum principle, $f^{n_k} \to \infty$ inside $\gamma$, contrary to Theorem 4.1.

It is really not hard to show that $\operatorname{ind}_0(f^n\gamma) \to \infty$, $n \to \infty$. ●

LEMMA 4.2. *Suppose that $f$ is bounded on some curve $\Gamma$ going to $\infty$. Then all the components of $F(f)$ are simply connected.*

PROOF. In the contrary case we consider a Jordan curve $\gamma$ that is not contractible in $D$. It follows from Lemma 4.1 that $f^n\gamma$ intersects $\Gamma$ for all sufficiently large $n$. Let $z_n \in f^n\gamma \cap \Gamma$. Then $fz_n \in f^{n+1}\gamma \to \infty$, $n \to \infty$, which contradicts the boundedness of $f$ on $\Gamma$. ●

We need a consequence of the well-known Harnack inequality ([7], Appendix).

LEMMA 4.3. *Let $V$ be an arbitrary domain, and $K$ a compact subset of $V$. If $h$ is a positive harmonic function in $V$, then $h(z_1) \leq ch(z_2)$ for every $z_1, z_2 \in K$, where the constant $c$ depends only on $V$ and $K$.*

THEOREM 4.3 (Baker [57]). *Let $f$ be a transcendental entire function. Then every unbounded component $D$ of $F(f)$ is simply connected.*

PROOF. Suppose that $D$ is multiply connected, and let $\gamma_0$ be a Jordan curve that is not contractible in $D$. It follows from Lemma 4.1 that for all sufficiently large $n$ the curves $\gamma_n = f^n\gamma_0$ intersect $D$, and hence are contained in $D$. Consequently, $D$ is invariant.

Let $V$ be a bounded domain containing the curves $\gamma_0$ and $\gamma_1$, and let $\overline{V} \subset D$. By Lemma 4.1, $|f^n z| > 1$, $z \in V$, for all sufficiently large $n$. Applying Lemma 4.3 to the functions $h_n(z) = \ln|f^n z|$ in the domain $V$, we get that for all $z_j \in \gamma_j$ $(j = 0, 1)$

$$|f^n z_1| \leq |f^n z_0|^c, \qquad n > n_0, \tag{4.1}$$

where $c$ does not depend on $n$.

On the other hand, since $\gamma_n \to \infty$, $\operatorname{ind}_0 \gamma_n > 0$, and $f$ is transcendental, it follows that for all sufficiently large $n$ there exists a point $\zeta \in \gamma_n$ such that $|f\zeta| > |\zeta|^c$. Let $\zeta = f^n z_0$, where $z_0 \in \gamma_0$, and $fz_0 = z_1 \in \gamma_1$. Then $|f^n z_1| = |f\zeta| > |\zeta|^c = |f^n z_0|^c$, contradicting the inequality (4.1). ●

COROLLARY 1. *If the set $F(f)$ for a transcendental entire function has an unbounded component $D$, then all the components of $F(f)$ are simply connected.*

PROOF. Let $V$ be a multiply connected, hence bounded, component of $F(f)$. It follows from Lemma 4.1 that $f^n V \cap D \neq \varnothing$ for all sufficiently large $n$. Consequently, $f^n V = D$, a contradiction. ●

COROLLARY 2. *The Julia set of a transcendental entire function cannot be totally disconnected.*

THEOREM 4.4. *The nonwandering components of the Fatou set of a transcendental entire function are simply connected.*

PROOF. Let $D$ be a nonwandering component of $F(f)$. If $D$ is unbounded, then it is simply connected by Theorem 4.3. If $D$ is bounded, then the iterates $f^n$ are uniformly bounded in $D$, and $D$ is simply connected by Lemma 4.1. ●

**2.2. A classification of periodic components.** Baker domains, Schröder domains, Boettcher domains, Leau domains, and Siegel disks are defined for an entire function just as for a rational function. According to Theorem 4.4, a transcendental entire function, like a polynomial, cannot have Arnol′d-Herman rings. However, there is a fundamentally new possibility for a transcendental entire function.

A component $D$ of the Fatou set of a transcendental entire function will be called a *Baker domain* if $f^n z \to \infty$ in $D$.

EXAMPLE 4.1 (Fatou [84]). $f\colon z \mapsto z + 1 + e^{-z}$. Obviously, the right half-plane $P = \{z\colon \operatorname{Re} z > 0\}$ is $f$-invariant, and $\operatorname{Re}(f^n z) \to +\infty$ for $z \in P$. Consequently, $P$ is contained in a Boettcher domain.

THEOREM 4.5. *Every periodic component of the Fatou set of a transcendental entire function is a Schröder domain, a Boettcher domain, a Leau domain, a Siegel disk, or a Baker domain.*

Let $f$ be an entire function. A point $a \in \mathbf{C}$ is said to be a *nonsingular* point (of the function $f^{-1}$) if it has a neighborhood $V$ such that $f\colon f^{-1}V \to V$ is an unbranched cover. The set of singular points is denoted by $\Sigma_f$. For nonlinear entire functions, $\Sigma_f \neq \varnothing$. Further, $\Sigma_f$ can be infinite, and can even coincide with the whole plane $\mathbf{C}$. Singular points are of the following types:

1) *$a$ is a critical value*;

2) *$a$ is an asymptotic value*, i.e., there exists a curve $\Gamma$ going to $\infty$ such that $fz \to a$ as $z \to \infty$ along $\Gamma$ (an important kind of singular point of this type is a *logarithmic branch point*, which has a simply connected neighborhood $V$ such that for some component $U$ of the set $f^{-1}V$ the mapping $f\colon U \to V \setminus \{a\}$ is a universal cover);

3) *limit points of singular points of types* 1) *and* 2).

See [37] and [93] for details about singular points of functions inverse to entire functions.

THEOREM 4.6. *A cycle of Schröder domains, Boettcher domains, or Leau domains contains at least one singular point. The boundary of a Siegel disk and of an arbitrary non-Siegel neutral cycle is contained in $\overline{\bigcup \omega(c)}$, where $c$ runs through $\Sigma_f$.*

A cycle of Baker domains does not necessarily contain singular points. This was shown independently by Herman [92] and the authors ([14], [79]).

EXAMPLE 4.2 [92]. Suppose that $f\colon z \mapsto \lambda z e^z$, where $\lambda = e^{2\pi i \Theta}$ and $\Theta$ satisfies the Siegel condition (1.2). Then $0$ is contained in a Siegel disk $D$. Raising the transformation $f\colon \mathbf{C}^* \to \mathbf{C}^*$ to the universal covering $\mathbf{C}$, we get an entire function $g\colon z \mapsto \ln\lambda + z + e^z$, for which $\exp^{-1} D$ is a Baker domain that is mapped univalently onto itself.

**2.3. Completely invariant components.** In this subsection we assume that $f$ is a transcendental entire function having a completely invariant component $D$ of the Fatou set. Obviously, $D$ is unbounded. By Corollary 1 to Theorem 4.3, all components of the Fatou set are simply connected.

THEOREM 4.7 ([56], [15]). *All singular points of the function $f^{-1}$ lie in $\overline{D}$. Further, the critical values and the logarithmic branch points lie in $D$.*

For a proof we need

THE GROSS THEOREM ([37], XI). *Let* $g(z)$ *be an element of* $f^{-1}$ *that is regular in a neighborhood of a point* $a$. *Then for almost all* $\theta \in [-\pi, \pi]$ *the element* $g(z)$ *can be continued analytically along the ray* $\{a+te^{i\theta} : 0 \le t < \infty\}$.

We can prove Theorem 4.7 only for critical values. Let $c$ be a critical point not in $D$, and let $a = fc$. By the Gross theorem, $a$ can be joined by a polygonal curve $\gamma(t)$, $0 \le t \le 1$, $\gamma(0) = a$, to a fixed point $b \in D$ in such a way that two analytic branches $g_i$ of $f^{-1}$ with $\lim_{t\to 0} g_i(\gamma(t)) = c$, $i = 1, 2$, are defined in a neighborhood of $\gamma$. Since $D$ is completely invariant, the curves $\Gamma_i = g_i\gamma$ end in $D$. We join their endpoints by a curve $\Delta$ lying in $D$, and we set $\delta = f\Delta$. Consider a bounded component $V$ of the complement of the curve $\Gamma_2^{-1}\Delta\Gamma_1$ such that $c \in \overline{V}$. Then $W = fV$ is a bounded domain such that: 1) $\partial W \subset \delta \cup \gamma$; and 2) $a \in W$. But $\delta$ is contained in the simply connected domain $D$, and hence $\operatorname{ind}_a \delta = 0$. Obviously, the situation is contradictory.

COROLLARY (Baker [56], [57]). a) *Let* $V$ *be a component of the Fatou set* $F(f)$ *different from* $D$. *Then* $f$ *is univalent on* $V$.

b) *A transcendental entire function cannot have two completely invariant components of the Fatou set.*

BAKER CONJECTURE [57]. *If a transcendental entire function has a completely invariant component* $D$, *then* $F(f) = D$.

The authors [15] proved this conjecture for the class $S$ of entire functions with finitely many singular points ($\operatorname{card}\Sigma_f < \infty$). This result follows from Theorem 4.7 and the description of the dynamics of the functions $f \in S$ (see §4).

## §3. Pathological examples

Wandering components of the Fatou set will be called *wandering domains.* The first example of such a domain was constructed by Baker ([53], [54]). This was at the same time of an example of a multiply connected component of the Fatou set. Simply connected wandering domains were constructed independently by Herman [91] and the authors ([14], [15], [79]).

EXAMPLE 4.3 (Herman). We consider the function $g\colon z \mapsto z - 1 + e^{-z}$. It has the superattracting fixed points $z_n = 2\pi i n$. Denote by $D_n$ the domain of immediate attraction of the point $z_n$. By Theorem 4.4, all the domains $D_n$ are simply connected.

The function $g$ commutes with the transformation $T\colon z \mapsto z + 2\pi i$. Consequently, $F(g)$ is $T$-invariant, and $TD_n = D_{n+1}$. Consider now the function $f\colon z \mapsto g(z) + 2\pi i$, $f = T \circ g$. We have that $fD_n = D_{n+1}$. We show that $J(f) = J(g)$. Let $z$ be an arbitrary repelling periodic point of $g$, $p$ its order,and $\lambda$ its multiplier. Then

$$f^{pn}z = T^{pn}(g^{pn}z) = T^{pn}z = z + (2\pi i)pn = O(n), \qquad n \to \infty,$$

and $(f^{pn})'(z) = (T^{pn})'(z)(g^{pn})'(z) = \lambda^n$. Therefore, for the spherical derivative we have that $|(f^{pn})'|/(1+|f^{pn}|^2) \to \infty$, $n \to \infty$. Consequently, the family $\{f^m\}$ is not normal in a neighborhood of $z$, i.e., $z \in J(f)$. By Theorem 4.1, $J(g) \subset J(f)$. The reverse inclusion can be proved similarly. Thus, the domains $D_n$ are wandering components of the Fatou set $F(f)$.

In the example of Herman all the mappings $f\colon D_n \to D_{n+1}$ are branched, and $f^n D_0 \to \infty$. The method used by the authors ([14], [15], [79]) permits the construction of examples that, though not elementary, are more subtle:

1. *An entire function $f$ having a wandering domain on which all the iterates $f^n$ are univalent.* Such a domain gives rise to an infinite-dimensional quasiconformal deformation of $f$ (see Chapter 2, §2.7).

2. *A wandering domain whose orbit has infinitely many limit points.* The authors do not know whether the orbit of such a domain is bounded.

3. *An entire function having infinitely many periodic domains of all types* (Schröder, Boettcher, Leau, Siegel, Baker) and infinitely many orbits of wandering domains of types 1 and 2.

4. *An entire function for which $J(f) \neq \mathbf{C}$, but* $\operatorname{meas} J(f) > 0$. *Here*

a) $\operatorname{meas}\{z \in J(f)\colon f^n z \to \infty,\ n \to \infty\} > 0$, *and*

b) $\operatorname{meas}\{z \in J(f)\colon \underline{\lim}_{n\to\infty} |f^n z| < \infty\} > 0$.

It is not known whether it is true that $\operatorname{meas}\{z \in J(f)\colon \overline{\lim}_{n\to\infty} |f^n z| < \infty\} > 0$.

McMullen recently discovered that property a) is possessed by all the transformations $z \mapsto \sin(az+b)$ [108].

5. *An entire function having an infinite-dimensional family of measurable invariant fields of lines on $J(f)$.* In this case the dynamics on the Julia set gives rise to an infinite-dimensional quasiconformal deformation of $f$ (see Chapter 2, §8).

It is possible to construct an entire function having all the properties 1–5 simultaneously.

## §4. The functions of the class $S$

**4.1. Simply connectedness of the components of the Fatou set.** *In the rest of this section it will be assumed that $f$ is transcendental, and the set $\Sigma_f$ of singular points of the function $f^{-1}$ is contained in the disk $\mathbf{U}_r$.* Denote by $H$ the exterior of this disk, and by $G$ the complete inverse image $f^{-1}H$. Let $V$ be some component of $G$.

LEMMA 4.4. *The domain $V$ is simply connected and bounded by a simple analytic curve going to $\infty$ at both endpoints.*

Since $|f(z)| = r$ on the curve $\Gamma = \partial V$, Lemma 4.2 gives us

LEMMA 4.5. *Under the assumptions made at the beginning of the subsection, all the components of the Fatou set $F(f)$ are simply connected.*

**4.2. Logarithmic change of variable.** We now describe a logarithmic change of variable that is one of the basic tools for investigating entire functions in the class under consideration in a neighborhood of $\infty$. This device (in a different circle of problems) goes back to Teichmüller [143].

The function $\varphi = \ln f$ maps $V$ conformally and univalently onto the half-plane $P = \{z\colon \operatorname{Re} z > \ln r\}$. Let $r$ be large enough so that $|f(0)| < r$. Then

$0 \notin G$, and the function exp is univalent on each component of the set $U = \exp^{-1} G$. We consider the commutative diagram

$$\begin{array}{ccc} U & \xrightarrow{f} & P \\ {\scriptstyle\exp}\downarrow & & \downarrow{\scriptstyle\exp} \\ G & \xrightarrow[f]{} & H \end{array} \qquad (4.2)$$

The function $\hat{f}$ maps each component $W$ of the set $U$ conformally and univalently onto the half-plane $P$. We consider the mapping $\Phi\colon P \to W$ inverse to $\hat{f}$. Let $z \in W$. The disk of radius $\operatorname{Re}\hat{f}(z) - \ln r$ about $\hat{f}(z)$ is contained in $P$. It follows from the Koebe 1/4 theorem that the set $W$ contains the disk of radius $\frac{1}{4}\Phi'(\hat{f}(z))(\operatorname{Re}\hat{f}(z) - \ln r)$. On the other hand, since exp is univalent on $W$, $W$ does not contain vertical segments of length $2\pi$. Consequently,

$$|\hat{f}'(z)| \geq \frac{1}{4\pi}(\operatorname{Re}\hat{f}(z) - \ln r). \qquad (4.3)$$

**4.3. The absence of Baker domains** ([14], [15]).

THEOREM 4.8. *Suppose that $f$ is transcendental, and the set $\Sigma_f$ of singular points is bounded. If $z \in F(f)$, then the orbit $\{f^n z\}_{n=0}^{\infty}$ cannot tend to $\infty$.*

PROOF. Suppose that $z \in F(f)$ and $z_n = f^n z \to \infty$. Then there exists a disk $B$ of radius $R$ about $z$ in which the sequence $\{f^n\}$ tends uniformly to $\infty$. Consequently, all the $B_n = f^n B$ from some point on are contained in $G$ (see the diagram in (4.2)). It can be assumed that $B_n \subset G$ for all $n$. Let $A$ be some component of the set $\exp^{-1} B$, and let $A_n = f^n A$. Then $\exp A_n = B_n$. Consequently, $A_n \subset U$, and $\operatorname{Re}\hat{f}^n \to \infty$ uniformly in $A$. Suppose that $\zeta \in A$, $\zeta_n = \hat{f}^n\zeta \in A_n$, and $R_n$ is the radius of the maximal disk about $\zeta_n$ that is inscribed in $A_n$. By the 1/4 theorem, $R_{n+1} \geq \frac{1}{4}R_n|\hat{f}'(\zeta_n)|$. Since $\operatorname{Re}\hat{f}(\zeta_n) \to +\infty$, it follows from (4.3) that $|\hat{f}'(\zeta_n)| \to \infty$. Consequently, $R_n \to \infty$. But then some domain $A_n \subset U$ contains a vertical segment of length $2\pi$, a contradiction. ●

COROLLARY. *If the entire function $f$ is transcendental and its set of singular points is bounded, then $F(f)$ does not contain Baker domains.*

**4.4. A complete picture of the dynamics on the Fatou set.** We say that an entire function $f$ belongs to the class $S_q$ if the function $f^{-1}$ has $q$ singular points $a_1, \dots, a_q$. In this case the points $a_j$ are also called *basic* points. The union of all the classes $S_q$ is denoted by $S$ (the letter $S$ is used in honor of Speiser, who apparently was the first to consider this class [37]). Obviously, the polynomials are contained in $S$. Examples of transcendental entire functions in $S$ are $e^z$, $\sin z$, and $\int_0^z P(\zeta)\exp Q(\zeta)\,d\zeta$ where $P$ and $Q$ are polynomials. The class $S$ is closed under compositions. The functions in the class $S$ play an important role in the general theory of meromorphic functions ([37], [143]), and from the point of view of dynamics they are free of the basic pathologies encountered with entire functions.

Entire functions $f$ and $g$ will be regarded as equivalent if there exist homeomorphisms $\varphi$ and $\psi$ of the plane $\mathbf{C}$ such that $\psi \circ g = f \circ \varphi$. The family of functions equivalent to $g$ is denoted by $M = M_g$. If $g \in S_q$, then $M \subset S_q$.

A complex-analytic manifold structure is introduced on $M$ in such a way that the mapping

$$M \times \mathbf{C} \to \mathbf{C}, \qquad (f, z) \mapsto f(z)$$

is analytic in both variables. The dimension of this manifold is equal to $q+2$. As local coordinates we can take the basic points of a function $f \in M$ and the values of this function at two points.

The following property of the manifold $M$ is obvious: *if $f_1$ and $f_2$ are topologically conjugate and $f_1 \in M$, then $f_2 \in M$.* After this remark the next result can be proved just like Theorem 2.17 of Sullivan (even more simply, because of Lemma 4.5).

THEOREM 4.9. *The Fatou set of an entire function in the class $S$ does not have wandering components.*

This theorem was proved by the authors ([14], [15]), independently by Baker [54] (for a more restricted class of functions), and somewhat later by Goldberg and Keen [87].

Theorems 4.5, 4.8, and 4.9 give us at once a complete description of the dynamics on the Fatou set of an entire function in the class $S$:

THEOREM 4.10 ([14], [15]). *Let $f \in S$. Then every orbit on the Fatou set is absorbed by one of the cycles of Schröder domains, Boettcher domains, Leau domains, or Siegel disks.*

COROLLARY. *If the orbits of all the basic points of a function $f \in S$ are absorbed by cycles or tend to $\infty$, then $J(f) = \mathbf{C}$.*

EXAMPLE 4.4 [58]. $f_w\colon z \mapsto wze^z$. This is a function of class $S_2$ with critical value $a_1 = -we^{-1}$ and with logarithmic branch point $a_2 = 0$, which is a fixed point. For a suitable choice of the parameter $w$ the orbit of the point $a_1$ is absorbed by a cycle. Consequently, $J(f_w) = \mathbf{C}$. Historically, this is the first example of an entire function with this property.

EXAMPLE 4.5. $J(\exp) = \mathbf{C}$ [119].

A function in the class $S$ has only finitely many nonrepelling cycles. This can be proved by the method of Fatou (see Chapter 2, §2.6) with account taken of the following fact, which is not obvious for transcendental entire functions.

LEMMA 4.6 ([15], [17]). *Let $f \in S$, let $\alpha(w)$ be a periodic point of the function $wf$ as an analytic multivalued function of the parameter $w$, and let $\lambda(w)$ be the multiplier of the point $\alpha(w)$. Then an arbitrary branch of the function $\lambda$ is nonconstant.*

This approach gives the estimate $2q$ for the number of nonrepelling cycles of a function in the class $S_q$. A better estimate can be obtained by the method of Shishikura.

THEOREM 4.11. *Let $f \in S_q$. In the notation of* 2.6 *in Chapter* 2,

$$N_A + N_L + N_I + N_S \leq q.$$

This estimate is apparently sharp in each family $M_f$.

**4.5. The measure of the Julia set.** We present an analogue of Theorem 3.12 for functions in the class $S$. An entire function $f$ is said to be a *function of finite order* if

$$\max_{|z|=r} |f(z)| \leq C \exp r^\rho$$

for some $\rho > 0$ and $C > 0$.

THEOREM 4.12 ([14], [15]). *Let $f$ be an entire function of finite order, and suppose that $f^{-1}$ has at least one logarithmic branch point. Then there exists an $R > 0$ such that $\underline{\lim}_{n\to\infty} |f^n z| \leq R$ for a.e. $z \in \mathbf{C}$. If it is assumed in addition that $f \in S$ and the orbits of all basic points in $J(f)$ are absorbed by cycles, then either $J(f) = \mathbf{C}$ or* $\operatorname{meas} J(f) = 0$.

Let $f\colon z \mapsto \sin(az + b)$, $a, b \in \mathbf{C}$. McMullen [108] showed that the set $\{z\colon f^n z \to \infty\}$ has positive measure. Of course, the parameters can be chosen so that the basic points of $f$ lie on $F(f)$. Consequently, the existence of a logarithmic branch point is essential for the validity of Theorem 4.12.

**4.6. Structural stability of entire functions of class $S$.** This theory is constructed according to the same scheme as for rational functions (see Chapter 2, §6). However, the transcendental case requires a certain analytical preparation whose main result we now formulate. A periodic point $\alpha(f)$ of a transformation $f$ will be regarded as a multivalued analytic function on $M$.

LEMMA 4.7 ([16], [17]). *The function $\alpha(f)$ has only algebraic singularities.*

We consider an analytic family $\mathfrak{M}$ of entire functions of class $S$, i.e., a complex submanifold of $M$.

THEOREM 4.13 ([16], [17]). *A function in general position in the family $\mathfrak{M}$ is structurally stable. The conjugating homeomorphism can be chosen to be quasiconformal.*

Just as in the rational case, there is a remarkable class among the $J$-stable entire functions: the functions satisfying axiom $A$. In this case axiom $A$ means that the orbits of the basic points tend to attracting cycles. An analogue of the Fatou conjecture is very likely for the family $M$ (Chapter 2, §6.4).

## 5.1. Exponential transformations

**5.1. The bifurcation diagram.** It is not hard to show that every transcendental function of class $S_1$ has the form $be^{cz} + a$. The investigation of iterates of such functions has a long history, going back to Euler. The popularity of this circle of problems increased sharply after the appearance of Misiurewicz' paper [119], in which he proved the property $J(\exp) = \mathbf{C}$, conjectured by Fatou [84].

Taking the quotient of the family $be^{cz} + a$ by the action of the affine group of conjugacies leads to the reduced family $z \mapsto e^{cz}$, $c \in \mathbf{C}^*$. We consider the family $f_a\colon z \mapsto e^z + a$, $a \in \mathbf{C}$, in order that the family not have singularities in the parameter plane. If the functions $f_a$ and $f_{a'}$ are affinely conjugate, then $a' = a + 2\pi i n$. The projection of the family $f_a$ on the reduced family has the form $c = e^a$. An immediate consequence of Theorems 4.10, 4.11, 4.6, and 4.12 is

THEOREM 4.14. *One of the following possibilities holds for the transformation $f_0\colon z \mapsto e^z + a$:*

*1) $f_a$ has a unique attracting cycle $\alpha = \{\alpha_k\}_{k=0}^{p-1}$. The Fatou set $F(f_a)$ coincides with the domain of attraction of this cycle. The Lebesgue measure of the Julia set $J(f_a)$ is equal to zero. The basic point $a$ is contained in the domain of immediate attraction of the cycle $\alpha$, but its orbit is not absorbed by this cycle.*

*2) $f_a$ has a unique neutral rational cycle $\alpha$. The remaining properties of $f_a$ are the same as in case 1).*

3) *$f_a$ has one cycle of Siegel disks. The boundaries of the Siegel disks are contained in* $\omega(a)$.([13])
4) $J(f_a) = \mathbf{C}$.

This theorem was proved independently by Baker and Rippon [59] (without the measure of the set $J(f_a)$) and the authors ([14], [15]). McMullen [108] showed that the Hausdorff dimension of the set $J(f_a)$ is equal to 2 for arbitrary $a$.

If $f_a$ has an attracting cycle, then one of the Schröder domains contains the left half-plane $\{z\colon \operatorname{Re} z < -N\}$ for large $N$. Consequently, all the components of $F(f_a)$ are unbounded. If the order of the attracting cycle is greater than 1, then there are infinitely many of these components. But if $f_a$ has an attracting fixed point, then $F(f_a)$ consists of a single completely invariant component. It can be shown that in this case $J(f_a)$ is a Cantor set of curves. Consequently, it contains points that are inaccessible from $F(f)$. However, as shown in [70], *a conformal mapping* $\mathbf{U} \to F$ *has radial limits everywhere on the boundary* $\partial\mathbf{U}$.

The set $W_1$ of points $a$ for which $f_a$ has an attracting fixed point is the domain bounded by the cycloid $a = it - e^{it}$, $-\infty < t < \infty$. This domain contains the real semi-axis $\{a\colon a < -1\}$. On the cycloid itself (in particular, at $a = -1$) the function has a neutral fixed point. The cases 2), 3), or 4) of Theorem 4.13 are realized according to its multiplier. We have that $f^n a \to \infty$ for $a > -1$, and hence $J(f_a) = \mathbf{C}$ (in particular, we get the Misiurewicz theorem for $a = 0$).

By Theorem 4.13, the open dense subset $R$ of the $a$-plane consists of structurally stable transformations. It is clear that these transformations must be of type 1) or 4) in Theorem 4.14. Can a function $f_a$ with $J(f_a) = \mathbf{C}$ be structurally stable? This question has not been answered, like the more general question formulated at the end of §4. An interesting result connected with this question was obtained by Devaney [69]: *the transformation* $z \mapsto e^z$ *is not structurally stable.*

According to the general terminology, the components of $R$ in which the functions $f_a$ have an attracting cycle are called *$A$-domains*. As shown in [15] and [59], the $A$-domains are simply connected and unbounded. *Is it true that the boundary of an $A$-domain is a simple curve?*

Just as in the case of the quadratic family (see Chapter 2, §7), the multiplier $\lambda$ of an attracting cycle can be regarded as a function in an arbitrary $A$-domain $W$. For this function the authors established an analogue of Theorem 2.39: *the function* $\lambda\colon W \to \mathbf{U}^*$ *is a universal cover.*

The bifurcation diagrams for the exponential family and the quadratic family have analogous structures. However, in the case of the exponential family it is not clear whether there is at least one tree of $A$-domains besides the one that grows from the domain $W_1$.

CONJECTURE. *The exponential family contains infinitely many trees of $A$-domains.*

**5.2. The topological and measurable dynamics of the transformation** $z \mapsto e^z$. As we have already mentioned, the extended investigation of the dynamics of the exponential began with the Misiurewicz theorem: $J(\exp) = \mathbf{C}$. It implies that the exponential is a mixing transformation, and hence the typical orbit in

([13])It is curious that the Siegel disks of the exponential function are unbounded (Herman [92]).

the Baire sense is dense in $J$. All cycles of the exponential are repelling and fill the plane densely. A more detailed study of the cycles of the exponential was undertaken in [72].

We now proceed to a description of the measurable dynamics of the exponential transformation. At present this question has been investigated very thoroughly. The effects discovered are in sharp contrast to the topological properties of the dynamics of the exponential described above, as well as to the known ergodic properties of rational endomorphisms. We begin with an unexpected result obtained independently by one of the authors [28] and Mary Rees [125]:

THEOREM 4.15. *The limit set $\omega(z)$ of almost every (with respect to Lebesgue measure) orbit $\{f^n z\}_{n=0}^{\infty}$ coincides with the orbit $\{f^n 0\}_{n=0}^{\infty}$ of zero.*

The following picture is concealed behind this formulation. Let $z$ be a typical point in the Lebesgue sense. Then at some moment its orbit comes close to zero, after which it moves for some time (how much?) along the orbit of zero. Then it goes away from that orbit and executes several jumps (how many? and how?) in the right half-plane. It then executes a jump into the left half-plane, from which it moves still closer to zero than originally by a single jump. After this the cycle repeats, except with greater amplitude.

Answers have been obtained to all the questions posed in parentheses [30]. For example, if $k_s$ is the time during which the orbit $\{f^n z\}_{n=0}^{\infty}$ moves along the trajectory of zero at the $s$th turn, then $k_s \sim 3.5s$, and the average time spent by the orbit in the right half-plane far from the trajectory of zero is equal to 1.

Further results have been obtained by Lyubich [30]. The first of them solves a problem posed by Sullivan [137]:

THEOREM 4.16. *The transformation $f\colon z \mapsto e^z$ is not ergodic with respect to the Lebesgue measure. Each of its ergodic components has zero measure.*

Thus, the exponential transformation has a continuum of ergodic components. The analogous fact for the transformation $z \mapsto \sin(az+b)$ was recently established by McMullen [108]. For contrast we mention that all the known rational endomorphisms for which $J(f) = \overline{\mathbf{C}}$ are ergodic with respect to Lebesgue measure (see Chapter 3, §2).

In Chapter 3 it was explained why it is important to look for invariant measures equivalent to Lebesgue measure.

THEOREM 4.17. *The transformation $f\colon z \mapsto e^z$ does not have an absolutely continuous invariant measure that is finite on compact sets.*

There obviously exists an absolutely continuous invariant measure whose density has a nonintegrable singularity only at the points $f^n 0$, $n \in \mathbf{N}$.

Further, the maximal quasiconformal deformation of the exponential is finite-dimensional (three-dimensional). Therefore, $f$ does not have wandering sets of positive measure on which all the iterates $f^n$ are injective [86] (cf. Chapter 3, §2.3). As shown by the following result, the last condition is essential.

THEOREM 4.18. *The transformation $f\colon z \mapsto e^z$ has a wandering set of positive measure.*

## BIBLIOGRAPHY

1. William Abikoff, *The real analytic theory of Teichmüller space*, Lecture Notes in Math., Vol. 820, Springer-Verlag, 1980.

2. Lars V. Ahlfors, *Lectures on quasiconformal mappings*, Van Nostrand, Princeton, N.J., 1966.

3. V. I. Arnol′d, *Supplementary chapters in the theory of ordinary differential equations*, "Nauka", Moscow, 1978; English transl., *Geometric methods in the theory of ordinary differential equations*, Springer-Verlag, 1982.

4. L. E. Boettcher, *The principal convergence laws for iterates and their applications to analysis*, Izv. Fiz.-Mat. Obshch. pri Imper. Kazanskom″Univ. **13** (1903), no. 1, 1–37; **14** (1904), nos. 3–4, 155–234. (Russian)

5. Patrick Billingsley, *Ergodic theory and information*, Wiley, 1965.

6. Rufus Bowen, *The methods of symbolic dynamics*, "Mir", Moscow, 1979. (Russian)(*)

7. M. Brelot, *Éléments de la théorie classique du potential*, 2nd ed., Centre de Documentation, Paris, 1961.

8. Georges Valiron, *Fonctions analytiques*, Presses Univ. France, Paris, 1954.

9. A. P. Veselov, *Integrable mappings and Lie algebras*, Dokl. Akad. Nauk SSSR **292** (1987), 1289–1291; English transl. in Soviet Math. Dokl. **35** (1987).

10. S. M. Voronin, *Analytic classification of germs of conformal mappings with linear part the identity*, Funktsional. Anal. i Prilozhen. **15** (1981), no. 1, 1–17; English transl. in Functional Anal. Appl. **15** (1981).

11. I. A. Golovinskiĭ, *Early history of analytic iterates and functional equations*, Istor. Mat. Issled. vyp. 25 (1980), 25–51. (Russian)

12. Adolf Hurwitz and R. Courant, *Vorlesungen über allgemeinen Funktionentheorie und elliptischen Funktionen*, 3rd ed., Springer-Verlag, 1929; reprint, Interscience, New York, 1944.

13. A. K. D′yudni, *Obtaining representations of very complicated mathematical objects with the help of a computer microscope*, V. Mire Nauki **1985**, no. 10, 80–87. (Russian)

14. A. È. Eremenko and M. Yu. Lyubich, *Iterates of entire functions*, Dokl. Akad. Nauk SSSR **279** (1984), 25–27; English transl. in Soviet Math. Dokl. **30** (1984).

15. ——, *Iterates of entire functions*, Preprint No. 6-84, Fiz.-Tekhn. Inst. Nizkikh Temperatur Akad. Nauk Ukr. SSR, Khar′kov, 1984. (Russian)

16. ——, *Structural stability in certain families of entire functions*, Funktsional. Anal. i Prilozhen. **19** (1985), no. 4, 86–87; English transl. in Functional Anal. Appl. **19** (1985).

17. ——, *Structural stability in certain families of entire functions*, Preprint No. 29-84; Fiz.-Tekhn. Inst. Nizkikh Temperatur Akad. Nauk Ukr. SSR, Khar′kov, 1984. (Russian)

18. Carl Ludwig Siegel, *Vorlesungen über Himmelsmechanik*, Springer-Verlag, 1956.

19. I. P. Kornfel'd, Ya. G. Sinaĭ, and S. V. Fomin, *Ergodic theory*, "Nauka", Moscow, 1980; English transl., Springer-Verlag, 1982.

20. S. L. Krushkal', B. N. Apanasov, and N. A. Gusevskii, *Kleinian groups and uniformization in examples and problems*, "Nauka", Novosibirsk, 1981; English transl., Amer. Math. Soc., Providence, R.I., 1986.

21. G. M. Levin, *On nonregular values of the parameter of a family of polynomial mappings*, Uspekhi Mat. Nauk **35** (1981), no. 6 (216), 219–220; English transl. in Russian Math. Surveys **35** (1981).

22. ——, *On arithmetic properties of a certain sequence of polynomials*, Uspekhi Mat. Nauk **34** (1988), no. 1 (259), 203–204; English transl. in Russian Math. Surveys **43** (1988).

(*)*Editor's note*. This volume consists of Russian translations of the following papers:

a) Rufus Bowen, *Equilibrium states and the ergodic theory of Anosov diffeomorphisms*, Lecture Notes in Math., Vol. 470, Springer-Verlag, Berlin and New York, 1975.

b) ——, *Markov partitions and minimal sets for Axiom A diffeomorphisms*, Amer. J. Math. **92** (1970), 907–917.

c) ——, *Symbolic dynamics for hyperbolic flows*, Amer. J. Math. **95** (1973), 429–459.

d) ——, *A horseshoe with positive measure*, Invent Math. **29** (1975), 203–204.

e) ——, *Topological entropy for noncompact sets*, Trans. Amer. Math. Soc. **184** (1973), 125–136.

f) Rufus Bowen and David Ruelle, *The ergodic theory of Axiom A flows*, Invent. Math. **29** (1975), 181–202.

It also contains the following paper:

g) V. M. Alekseev and M. V. Jakobson, *Symbolic dynamics and hyperbolic dynamical systems* (Russian pp. 196–240); English transl., Phys. Rep. **75** (1981), 287–325.

23. M. Yu. Lyubich, *Entropy of analytic endomorphisms of the Riemann sphere*, Funktsional. Anal. i Prilozhen. **15** (1981), no. 4, 83–84; English transl. in Functional Anal. Appl. **15** (1981).

24. ——, *On the measure of maximal entropy of a rational endomorphism of the Riemann sphere*, Funktsional Anal. i Prilozhen. **16** (1982), no. 4, 78–79; English transl. in Functional Anal. Appl. **16** (1982).

25. ——, *On the typical behavior of the trajectories of a rational mapping of the sphere*, Dokl. Akad. Nauk SSSR **268** (1983), 29–32; English transl. in Soviet Math. Dokl. **27** (1983).

26. ——, *Some typical properties of the dynamics of rational mappings*, Uspekhi Mat. Nauk **38** (1983), no 5 (233), 197–198; English transl. in Russian Math. Surveys **38** (1983).

27. ——, *An investigation of stability for the dynamics of rational functions*, Teor. Funktsiĭ Funktsional. Anal. i Prilozhen. vyp. 42 (1984), 72–91. (Russian)

28. ——, *Measurable dynamics of exponentials*, Dokl. Akad. Nauk SSSR **292** (1987), 1301–1304; English transl. in Soviet Math. Dokl. **35** (1987).

29. ——, *Dynamics of rational transformations: a topological picture*, Uspekhi Mat. Nauk **41** (1986), no. 4 (250), 35–95; English transl. in Russian Math. Surveys **41** (1986).

30. ——, *Measurable dynamics of exponentials*, Sibirsk Mat. Zh. **28** (1987), no. 5, 111–127; English transl. in Siberian Math. J. **28** (1987).

31. M. Yu. Lyubich and Yu. I. Lyubich, *The Perron-Frobenius theory for almost periodic operators and representations of semigroups*, Teor. Funktsiĭ Funktsional. Anal. i Prilozhen. vyp. 46 (1986), 54–72. (Russian)

32. Yu. I. Lyubich, *A remark about stability of complex dynamical systems*, Izv. Vyssh. Uchebn. Zaved. Mat. **1983**, no. 10 (257), 49–50; English transl. in Soviet Math. (Iz. VUZ) **1983**.

33. ——, *Introduction to the theory of Banach representations of groups*, "Vishcha Shkola", Khar′kov, 1985; English transl., Birkhäuser, Basel, 1988. (Russian)

34. N. G. Makarov, *Harmonic measure and Hausdorff measure*, Dokl. Akad. Nauk SSSR **280** (1985), 545–548; English transl. in Soviet Math. Dokl. **81** (1985).

35. P. M. Makienko, *Iteration of analytic functions on* $\mathbf{C}^*$, Dokl. Akad. Nauk SSSR **297** (1987), 35–37; English transl. in Soviet Math. Dokl. **36** (1988).

36. Paul Montel, *Leçons sur les familles normales de fonctions analytiques et leurs applications*, Gauthier-Villars, Paris, 1927.

37. Rolf Nevanlinna, *Eindeutige analytische Funktionen*, Springer-Verlag, 1936.

38. Zbigniew Nitecki, *Differentiable dynamics*, MIT Press, Cambridge, Mass., 1971; the 1975 Russian transl. ("Mir", Moscow) has an appendix by A. B. Katok, *Local properties of hyperbolic sets*, pp. 214–232.

39. Ya. B. Pesin, *Lyapunov characteristic exponents and smooth ergodic theory*, Uspekhi Mat. Nauk **32** (1977), no. 4 (196), 55–111; English transl. in Russian Math. Surveys **32** (1977).

40. V. A. Rokhlin, *On basic concepts of measure theory*, Mat. Sb. **67** (**109**) (1949), 109–150. (Russian)

41. ——, *Exact endomorphisms of Lebesgue spaces*, Izv. Akad. Nauk SSSR Ser. Mat. **25** (1961), 499–530; English transl. in Amer. Math. Soc. Trans. (2) **39** (1964).

42. Ya. G. Sinaĭ, *Gibbs measures in ergodic theory*, Uspekhi Mat. Nauk **27** (1972), no. 4 (166), 21–64; English transl. in Russian Math. Surveys **27** (1972).

43. Itogi Nauki i Tekhniki: Sovremennye Problemy Mat., Fundamental. Napravleniya, vol. 1, Dinamicheskie sistemy, vol. 1, VINITI, Moscow, 1985; English transl., Encyclopedia of Math. Sci., vol. 1 (Dynamical systems, 1), Springer-Verlag, 1987.

44. Itogi Nauki i Tekhniki: Sovremennye Problemy Mat., Fundamental. Napravleniya, vol. 2, Dinamicheskie sistemy, vol. 2, VINITI, Moscow, 1985; English transl., Encyclopedia of Math. Sci., vol. 2 (Dynamical systems, 2), Springer-Verlag, 1988.

45. S. Stoïlow, *Leçons sur les principes topologiques de la théorie des fonctions analytiques*, 2nd ed., Gauthier-Villars, Paris, 1956.

46. Paul R. Halmos, *Lectures on ergodic theory*, Math. Soc. Japan, Tokyo, 1956; reprint, Chelsea, 1960.

47. W. K. Hayman, *Meromorphic functions*, Clarendon Press, Oxford, 1964.

48. M. V. Yakobson, *The structure of polynomial mappings on a singular set*, Mat. Sb. **77**(**119**) (1968), 105–124; English transl. in Math. USSR Sb. **6** (1968).

49. ——, *On the question of classifying polynomial endomorphisms of the plane*, Mat. Sb. **80** (**122**) (1969), 365–387; English transl. in Math. USSR Sb. **9** (1969).

50. ——, *On the question of topological classification of rational mappings of the Riemann sphere*, Uspekhi Mat. Nauk **28** (1973), no. 2 (170), 247–248. (Russian)

51. ——, *On the boundaries of certain domains of normality for rational mappings*, Uspekhi Mat. Nauk **39** (1984), no. 6 (240), 211–212; English transl. in Russian Math. Surveys **39** (1984).

52. W. Abikoff, *The residual limit set of Kleinian groups*, Acta Math. nos. 1 and 2, **130** (1973), 127–144.

53. I. N. Baker, *Multiply connected domains of normality in iteration theory*, Math. Z. **81** (1963), 206–214.

54. ——, *Wandering domains in the iteration of entire functions*, Proc. London Math. Soc. **49** (1984), 563–576.

55. ——, *Repulsive fixpoints of entire functions*, Math. Z. **104** (1968), 252–256.

56. ——, *Completely invariant domains of entire functions*, Math. essays dedicated to A. J. MacIntyre, Ohio Univ. Press, Athens, Ohio, 1970.

57. ——, *The domains of normality of an entire function*, Ann. Acad. Sci. Fenn. Ser. A I Math. **1** (1975), 277–283.

58. ——, *Limit functions and sets of non-normality in iteration theory*, Ann. Akad. Sci. Fenn. Ser. A I Math. **467** (1970).

59. I. N. Baker and P. J. Rippon, *Iteration of exponential functions*, Ann. Acad. Sci. Fenn. Ser. A I Math. **9** (1984), 49–77.

60. L. Bers and H. L. Royden, *Holomorphic families of injections*, Acta Math. (1986), nos. 3 and 4, 259–286.

61. P. Blanchard, *Complex analytic dynamics on the Riemann sphere*, Bull. Amer. Math. Soc. (N.S.) **11** (1984), 85–141.

62. R. Bowen, *Hausdorff dimension of quasi-circles*, Inst. Hautes Études Sci. Publ. Math. **50** (1979).

63. H. Brolin, *Invariant sets under iteration of rational functions*, Ark. Mat. **6** (1965), 103–144.

64. L. Carleson, *The support of harmonic measure for sets of Cantor type*, Ann. Acad. Sci. Fenn. Ser. A I Math. **10** (1985), 113–123.

65. P. Collet and J. Eckmann, *Iterated maps of the interval as dynamical systems*, Birkhäuser, Basel, 1980.

66. C. Cowen, *Iteration and the solution of functional equations for functions analytic in the unit disk*, Trans. Amer. Math. Soc. **265** (1981), 69–95.

67. J. Curry, L. Garnett, and D. Sullivan, *On the iteration of rational function: computer experiments with Newton's method*, Comm. Math. Phys. **91** (1983), 267–277.

68. F. M. Dekking, *Recurrent sets*, Adv. in Math. **44** (1982), 78–104.

69. R. L. Devaney, *Structural instability of* $\exp(z)$, Proc. Amer. Math. Soc. **94** (1985), 545–548.

70. R. L. Devaney and L. R. Goldberg, *Uniformization of attracting basins for exponential maps*, preprint, Math. Sci. Res. Inst., Berkeley, California, 1986.

71. R. L. Devaney, L. R. Goldberg, and J. Hubbard, *Dynamical approximation to the exponential map by polynomials*, Preprint, Math. Sci. Res. Inst., Berkeley, California, 1986.

72. R. L. Devaney and M. Krych, *Dynamics of* $\operatorname{Exp}(z)$, Ergodic Theory Dynamical Systems **4** (1984), 35–52.

73. A. Douady, *Systèmes dynamiques holomorphes*, Seminaire Bourbaki, Exposé 599, 1982/1983.

74. ——, *Surgery on holomorphic mappings*, Proc. Internat. Congr. Math., Amer. Math. Soc., Providence, R.I., 1987.

75. A. Douady and J. Hubbard, *Itération des polynômes quadratiques complexes*, C. R. Acad. Sci. Sér. I Math. **294** (1982), 123–126.

76. ——, *Étude dynamique des polynômes complexes*. I, II, Publ. Math. Orsay, 84-02, 85-02, Univ. Paris XI, Orsay, 1984, 1985.

77. ——, *A proof of Thurston's topological characterization of rational functions*, Preprint, Inst. Mittag-Leffler, Report No. 2, 1985.

78. ——, *On the dynamics of polynomial-like mappings*, Ann. Sci. École Norm. Sup. (4) **18** (1985), 287–343.

79. A. È. Eremenko and M. Yu. Lyubich, *Examples of entire functions with pathological dynamics*, J. London Math. Soc. **36** (1987), 458–468.

80. P. Fatou, *Sur les équations fonctionnelles*, Bull. Soc. Math. France **47** (1919), 161–271.

81. ——, *Sur les équations fonctionnelles*, Bull. Soc. Math. France **48** (1920), 33–94.

82. ——, *Sur les équations fonctionnelles*, Bull. Soc. Math. France **48** (1920), 208–314.

83. ——, *Sur les fonctions qui admettent plusieurs théorémes de multiplication*, C. R. Acad. Sci. Sér I Math. **173** (1921), 571–573.

84. ——, *Sur l'itérations des fonctions transcendantes entières*, Acta Math. **47** (1926), 337–370.

85. A. Freire, A. Lopes, and R. Mañé, *An invariant measure for rational maps*, Bol. Soc. Brasil Mat. **14** (1983), 45–62.

86. E. Ghys, L. Goldberg, and D. Sullivan, *On the measurable dynamics of* $z \mapsto e^z$, Ergodic Theory Dynamical Systems **5** (1975), 329–335.

87. L. R. Goldberg and L. Keen, *A finiteness theorem for a dynamical class of entire functions*, Ergodic Theory Dynamical Systems **6** (1986), 183–192.

88. M. Gromov, *On the entropy of holomorphic maps*, Preprint, Inst. Hautes Études Sci., Bures-sur-Yvette, 1980.

89. ——, *Entropy, homology and semialgebraic geometry* (*after Y. Yomdin*), Seminaire Bourbaki, Exposé 663, 1985/1986, pp. 225–240.

90. J. Guckenheimer, *Endomorphisms of the Riemann sphere*, Global Analysis, Proc. Sympos. Pure Math., Vol. 48, Amer. Math. Soc., Providence, R.I., 1970, pp. 95–124.

91. M. R. Herman, *Exemples de fractions rationnelles ayant une orbite dénse sur la sphère de Riemann*, Bull. Soc. Math. France **112** (1984), 93–142.

92. ——, *Are there critical points on the boundaries of singular domains*? Comm. Math. Phys. **99** (1985), 593–612.

93. F. Iversen, *Recherches sur les fonctions inverses des fonctions méromorphes*, Thèse de Helsingfors, 1914.

94. G. Julia, *Mémoire sűr l'itération des fonctions rationnelles*, J. Math. Pures Appl. **8** (1918), 47–245.

95. ——, *Sur une classe d'équations fonctionnelles*, Ann. Sci. École Norm. Sup. **40** (1923), 13–150.

96. ——, *Mémoire sur la permutabilité des fractions rationnelles*, Ann. Sci. École Norm. Sup. **39** (1922), 131–215.

97. I. Jungreis, *The uniformization of the complement of the Mandelbrot set*, Duke Math. J. **52** (1985), 935–938.

98. K. Krzyzewsky, *On connection between expanding mappings and Markov chains*, Bull. Acad. Polon. Sci. Sér. Sci. Math. **19** (1971), 291–293.

99. K. Krzyzewsky and K. Szlenk, *On invariant measures for expanding differentiable mappings*, Studia Math. **33** (1969), 83–92.

100. F. Ledrappier, *Some properties of absolutely continuous invariant measures on an interval*, Ergodic Theory Dynamical Systems **1** (1981), 77–94.

101. ——, *Quelques propriétés ergodiques des applications rationnelles*, C. R. Acad. Sci. Sér. I Math. **299** (1984), 37–40.

102. M. Yu. Lyubich, *Entropy properties of rational endomorphisms of the Riemann sphere*, Ergodic Theory Dynamical Systems **3** (1983), 351–386.

103. A. Lopes, *Equilibrium measures for rational maps*, Ergodic Theory Dynamical Systems **6** (1986), 393–399.

104. R. Mañé, *On the instability of Herman rings*, Invent Math. **81** (1985), 459–472.

105. N. G. Makarov, *On the harmonic measure of the snowflake*, LOMI Preprint, E-4-86, Leningrad, 1986.

106. N. G. Makarov and A. L. Volberg, *On the harmonic measure of discontinuous fractals*, LOMI Preprint, E-6-86, Leningrad, 1986.

107. C. McMullen, *Families of rational maps and iterative root-finding algorithms*, Ann. of Math. **125** (1987), 467–494.

108. ——, *Area and Hausdorff dimension of Julia sets of entire functions*, Trans. Amer. Math. Soc. **300** (1987), 339–342.

109. ——, Automorphisms of rational maps, Preprint, Math. Sci. Res. Inst. Berkeley, California, 1986.

110. ——, *Braiding of the attractor and the failure of iterative algorithms*, Invent. Math. **91** (1988), 259–272.

111. B. Mandelbrot, *Fractal aspects of the iteration of* $z \mapsto \lambda z(1-z)$ *for complex* $z$ *and* $\lambda$, Ann. New York Acad. Sci. **357** (1980), 249–259.

112. ——, *The fractal geometry of nature*, Freeman, New York, 1983.

113. R. Mañé, *On the uniqueness of the maximising measure for rational maps*, Bol. Soc. Brasil Mat. **14** (1983), 27–43.

114. ——, *On the Bernoulli property for rational maps*, Ergodic Theory Dynamical Systems **5** (1985), 71–88.

115. R. Mañé, P. Sad, and D. Sullivan, *On the dynamics of rational maps*, Ann. Sci. École Norm. Sup. (4) **16** (1983), 193–217.

116. A. Manning, *The dimension of the maximal measure for a polynomial map*, Ann. of Math. (2) **119** (1984), 425–430.

117. ——, *How to be sure of solving a complex polynomial using Newton's method*, Preprint, Math. Inst. Univ. of Warwick, 1986.

118. N. F. G. Martin, *On finite Blaschke products whose restrictions to the unit circle are exact endomorphisms*, Bull. London Math. Soc. **15** (1983), 343–348.

119. M. Misiurewicz, *On iterates of* $e^z$, Ergodic Theory Dynamical Systems **1** (1981), 103–106.

120. M. Misiurewicz and F. Przytycki, *Topological entropy and degree of smooth mappings*, Bull. Acad. Polon. Sci. Sér. Sci. Math. **25** (1977), 573–574.

121. Ch. Pommerenke, *On conformal mapping and iteration of rational functions*, Complex Variables Theory Appl. **5** (1986), 117–126.

122. F. Przytycki, *Riemann map and holomorphic dynamics*, Invent. Math. **85** (1986), 439–455.

123. F. Przytycki, M. Urbanski, and A. Zdunik, *Harmonic, Gibbs and Hausdorff measures on repellers for holomorphic maps*, Preprint, Math. Inst. Univ. Warwick, 1986.

124. M. Rees, *Positive measure sets of ergodic rational maps*, Ann. Sci. École Norm. Sup. (4) **19** (1986), 383–407.

125. ——, *The exponential map is not recurrent*, Math. Z. **191** (1986), 593–598.

126. J. F. Ritt, *On the iteration of rational functions*, Trans. Amer. Math. Soc. **21** (1920), 348–356.

127. ——, *Periodic functions with a multiplication theorem*, Trans. Amer. Math. Soc. **23** (1922), 16–25.

128. ——, *Permutable rational functions*, Trans. Amer. Math. Soc. **25** (1923), 399–448.

129. H. L. Royden, *Automorphisms and isometries of Teichmüller spaces*, Advances in the Theory of Riemann Surfaces, Princeton Univ. Press, Princeton, N.J., 1971, pp. 369–383.

130. D. Ruelle, *Thermodynamic formalism*, Addison-Wesley, London, 1978.

131. ——, *An inequality for the entropy of differentiable maps*, Bol. Soc. Brasil Mat. **9** (1978), 83–87.

132. ——, *Repellers for real analytic maps*, Ergodic Theory Dynamical Systems **2** (1982), 99–107.

133. M. Shub and D. Sullivan, *Expanding endomorphisms of the circle revisited*, Ergodic Theory Dynamical Systems **5** (1985), 285–289.

134. S. Smale, *On the efficiency of algorithms of analysis*, Bull. Amer. Math. Soc. (N.S.) **13** (1985), 87–121.

135. D. Sullivan, *The ergodic theory at infinity of a discrete group of hyperbolic isometries*, (1978 Stony Brook Conference), Ann. of Math. Studies, Vol. 97, Princeton Univ. Press, Princeton, N.J., 1981, pp. 465–497.

136. ——, *Itération des fonctions analytiques complexes*, C. R. Acad. Sci. Sér. I Math. **294** (1982), 301–304.

137. ——, *Seminar on conformal and hyperbolic geometry*, Preprint, M/82/12, Inst. Hautes Études Sci., Bures-sur-Yvette, 1982.

138. ——, *Conformal dynamical systems*, Lecture Notes in Math., Vol. 1007, Springer, 1983, pp. 725–752.

139. ——, *Quasi conformal homeomorphisms and dynamics.* I. *Solution of Fatou-Julia problem on wandering domains*, Ann. of Math. (2) **122** (1985), 401–418.

140. ——, *Quasi conformal homeomorphisms and dynamics.* III. *Topological conjugacy classes of analytic endomorphisms*, Preprint, M/83/1, Inst. Hautes Études Sci., Bures-sur-Yvette, 1983.

141. D. Sullivan and W. Thurston, *Extending holomorphic motions*, Acta Math. (1986), 243–257.

142. W. Thurston, *On the combinatorics of iterated rational maps*, Preprint, Princeton Univ. and Inst. Adv. Study, 1985.

143. O. Teichmüller, *Eine Umkehrung des zweiten Hauptsatzes der Wertverteilungslehre*, Deutsch. Math. **2** (1937), 96–107.

144. P. Walters, *Invariant measures and equilibrium states for some mappings which expand distances*, Trans. Amer. Math. Soc. **236** (1978), 121–153.

145. J.-C. Yoccoz, *Linéarization des germes de difféomorphismes holomorphes de* $(\mathbf{C}, 0)$, C. R. Acad. Sci. Ser. I Math. **306** (1988).

146. A. Zdunik, *Hausdorff dimension of maximal measure for rational maps*, Preprint, Inst. Math. Polish Acad. Sci., Warsaw, 1987.

## Supplementary Bibliography

147. A. D. Bryuno, *Convergence of transformations of differential equations to normal forms*, Dokl. Akad. Nauk SSSR **165** (1965), 987–989; English transl. in Soviet Math. Dokl. **6** (1965).

148. A. Douady, *Disques de Siegel et anneaux de Herman*, Asterisque **152/153** (1987), 151–272.
149. M. Herman, *Recent results and some open questions on Siegel's linearization theorem*, Preprint. Paris, 1986.
150. J. Kotus, *Iterated holomorphic maps on punctured plane*, Preprint, no. 358, Inst. Math. Polish Acad. Sci., Warsaw, 1986.
151. H.-O. Peitgen and P. H. Richter, *The beauty of fractals*, Springer, Berlin, 1986.
152. F. Przytycki, *On Ω-stability and structural stability of endomorphisms satisfying Axiom A*, Studia Math. **60** (1977), 61–77.
153. M. Shishikura, *On the quasiconformal surgery of rational functions*, Ann. Sci. École Norm. Sup. (4) **20** (1987), 1–29.
154. D. Sullivan, *Quasi conformal homeomorphisms and dynamics*. II. *Structural stability implies hyperbolicity for Kleinian groups*, Acta Math. **155** (1985), 243–260.
155. ——, *Quasiconformal homeomorphisms in dynamics, topology and geometry*, Proc. Internat. Congr. Math., Amer. Math. Soc., Providence, R.I., 1987.

Physics-Engineering Institute of Low Temperatures
Academy of Sciences of the Ukrainian SSR
Khar′kov
Arctic and Antarctic Scientific Research Institute
Leningrad

Received 5/DEC/88

Translated by H. H. McFADEN