Meromorphic functions of one complex variable. A survey

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Abstract

This is an appendix to the English translation (to be published by AMS) of the book by A. A. Goldberg and I. V. Ostrovskii, Distribution of values of meromorphic functions, Moscow, Nauka, 1970. In this appendix we survey the results obtained on the topics of the book after 1970.

The literature on meromorphic functions[1] is very large. There is a comprehensive survey [62] that contains everything that was reviewed on the topic in the Soviet "Referativnyi Zhurnal" in 1953–1970, and a later large survey [67]. More recent surveys [30], [80] and [48] are shorter and have narrower scope.

Some books on specific topics in the theory of meromorphic functions published after 1970 are [18], [20], [79], [100], [137], [134], [138], [162], [165]. A survey of the fast developing subject of iteration of meromorphic functions is [7].

Here we give a short survey of some results which are closely related to the problems considered in this book. Thus we do not include many important topics, like geometric theory of meromorphic functions, iteration, composition, differential and functional equations, normal families, Borel and Julia directions, uniqueness theorems, and most regrettably, holomorphic curves and quasiregular mappings.

Chapter I.

Mokhon'ko [121] proved the following generalization of (6.29), Ch. I. *Let $R(w,z)$ be a rational function of w whose coefficients are meromorphic functions $h(z)$ satisfying $T(r,h) = O(\phi(r))$, where ϕ is a fixed positive increasing*

[1] If the domain is not specified explicitly, we mean meromorphic functions in **C**.

function on $[0,\infty)$. *Then for every meromorphic function* f *we have*

$$T(r, R(f(z), z)) = \deg_w R\, T(r, f) + O(\phi(r)).$$

This is a purely algebraic result; its proof uses only properties (6.5), (6.6), (6.8′) and the property $T(r, f^2) = 2T(r, f)$.

Another result in the same direction is due to Eremenko [39]. *Let* $F(u, v, z)$ *be a polynomial in* u *and* v *whose coefficients are meromorphic functions* h *of* z *satisfying* $T(r, h) = O(\phi(r))$, *where* ϕ *is a function as above, and assume that* F *is irreducible over the algebraic closure of the field of meromorphic functions. If* f *and* g *are meromorphic functions satisfying*

$$F(f(z), g(z), z) \equiv 0, \tag{1}$$

then

$$\deg_u F\, T(r, f) = (\deg_v F + o(1))T(r, g) + O(\phi(r)).$$

Both theorems have applications in the analytic theory of differential equations.

Vojta [153] noticed a formal analogy between the definition of the Nevanlinna characteristic and the definition of *height* in number theory. This analogy extends quite far, and it has been a source of many interesting results and conjectures in the recent years. For example, under Vojta's analogy, Jensen's formula corresponds to the fundamental theorem of arithmetic, while the second main theorem without the ramification term corresponds to the Thue–Siegel–Roth theorem on Diophantine approximation. The full second main Theorem corresponds to the famous unproved conjecture in number theory which is known as the "*abc*-conjecture". The analogy extends to the multi-dimensional generalizations of Nevanlinna theory, where it has been especially fruitful [137, 101]. As an example of the influence of Vojta's analogy on the one-dimensional value distribution theory, we mention the recent precise results on the error term in the second main theorem (see comments to Chapter III).

Analogies between function theory and number theory were noticed before Vojta, see, for example [126]. Osgood was the first to view the second main theorem as an analogue of the Thue–Siegel Roth theorem [129, 130, 131]. See also [69, 77].

Formula (2.6), Ch. I is the basis of the so-called "Fourier method" in the theory of entire and meromorphic functions, which was developed in the work of Rubel [138], Taylor, Miles and others. The Fourier method is considered as a substitute for the Weierstrass representation, which is more effective in certain cases. One of the main results achieved with this technique is the theorem of Miles [114] on the quotient representation of meromorphic functions: *Every meromorphic function f can be written as $f = g_1/g_0$, where g_j are entire functions, possibly with common zeros, satisfying*

$$T(r, g_j) \leq AT(Br, f), \quad j = 0, 1,$$

where A and B are absolute constants (independent of f). Moreover, for every $B > 1$ there exists A such that the above statement holds for every f. Simple examples show that one cannot in general take $B = 1$. This result was improved by Khabibullin [97]: *Let f be a meromorphic function and $\epsilon > 0$ a non-increasing convex function on $[0, \infty)$. Then there esists a representation $f = g_1/g_0$ such that g_j are entire functions, and*

$$\ln(|g_1(z)| + |g_0(z)|) \leq \frac{A_\epsilon}{\epsilon(|z|)} T((1 + \epsilon(|z|))|z|, f) + B_\epsilon.$$

Khabibullin also extended Miles's theorem to meromorphic functions in $\mathbf{C}^n$. A survey of his results is [98].

The Fourier method is also one of the main tools for the study of functions which satisfy various restrictions on the arguments of a-points (see comments to Chapter VI). A recent book on the Fourier method is [99].

Chapter II.

Various properties of increasing functions on $[0, \infty)$ play an important role in the theory of meromorphic functions. By 1970 it became clear that the lower order λ is at least as important as the order ρ. The main development since 1970 was "localizing" the notion of order.

Let Φ be an unbounded increasing function. A sequence $r_k \to \infty$ is called a sequence of *Pólya peaks* (of the first kind) of order μ if the inequalities

$$\Phi(rr_k) \leq (1 + \epsilon_k) r^\mu \Phi(r_k), \quad \epsilon_k < r < \epsilon_k^{-1}$$

hold with some $\epsilon_k \to 0$. Pólya peaks were formally introduced by Edrei [33], though they were used by Edrei and Fuchs already in 1963. Pólya peaks of order μ exist for all μ in certain interval $[\lambda^*, \rho^*]$, which contains the interval $[\lambda, \rho]$. The endpoints λ^* and ρ^* of this interval are called the order and lower order in the sense of Pólya, respectively. The following formulas were given in [31]:

$$\rho^* = \sup\{p\colon \limsup_{x,A\to\infty} g(Ax)/A^p g(x) = \infty\}$$

and

$$\lambda^* = \inf\{p\colon \liminf_{x,A\to\infty} g(Ax)/A^p g(x) = 0\}.$$

Most contemporary results on functions of finite lower order use Pólya peaks.

Chapter III.

Milloux' inequality (Ch. III, Theorem 2.4) has led to a rich vein of results developing the value distribution properties of meromorphic functions and their derivatives, in which a decisive role has been played by the paper [71] of Hayman.

Perhaps the most striking of the many results from [71] is *Hayman's alternative* (Ch. III, Theorem 2.6): if a function f meromorphic in the plane omits a finite value a, and its kth derivative $f^{(k)}$, for some $k \geq 1$, omits a finite non-zero value b, then f is constant. Two principal questions arising in connection with Hayman's alternative are: (i) whether a version of Hayman's main inequality (Ch. III, (2.23)) holds with $N(r, 1/(f-a))$ replaced by $\overline{N}(r, 1/(f-a))$; (ii) whether $f^{(k)}$ can be replaced by a more general term, such as a linear differential polynomial

$$F = L[f] = f^{(k)} + a_{k-1} f^{(k-1)} + \ldots + a_0 f, \tag{2}$$

with suitable coefficients a_j of small growth compared to f. A positive answer to (i) was given by Chen [17]. Question (ii) was answered affirmatively in [102], although there do exist exceptional functions f, which may be determined from the a_j, for which f and $F-1$ have no zeros. A unified approach to the questions from [17, 102] may be found in [14]. It was shown further in [19, 82] that if P is a non-constant differential polynomial in f, all of whose terms have degree at least 2 in f and its derivatives, then a version of Hayman's inequality holds with $f^{(k)}$ replaced by P, and with $\overline{N}$ counting functions.

Question (i) is related to the issue of whether $f-a$ and $f^{(k)}-b$ ($k \geq 1, b \neq 0$) can both fail to have simple zeros, in analogy with the sharp result that a nonconstant meromorphic function cannot be completely branched over five distinct values. It has recently been shown [128] using normal family methods that if f is transcendental and meromorphic in the plane with only multiple zeros then f' takes every finite non-zero value infinitely often (see also [14, 155]).

The obvious example $f(z) = e^z$ shows that a transcendental entire function f may have the property that f and all its derivatives omit 0: thus the condition $b \neq 0$ is necessary in Hayman's alternative. However, Hayman showed in [71] (see Ch. III, Theorem 2.7) that if f is an entire function such that ff'' has no zeros then $f(z) = \exp(Az + B)$ with A, B constants: this follows from applying (Ch. III, Theorem 2.6) to f/f'. Clunie [21] established the corresponding result with f'' replaced by a higher derivative $f^{(k)}$, which on combination with Hayman's theorem on ff'' significantly improved earlier results of Pólya and others on the zeros of entire functions and their derivatives [24, 135, 140, 141]. The *Tumura-Clunie method*, as developed in [21], shows that if $\Psi_k[g]$ is a differential polynomial of form $\Psi_k[g] = g^k + P_{k-1}[g]$, where $P_{k-1}[g]$ is a polynomial in g and its derivatives of total degree at most $k - 1$, and with coefficients of small growth compared to g, and if g has few poles and $\Psi_k[g]$ has few zeros, then $\Psi_k[g]$ admits a simple factorisation. The application to zeros of $ff^{(k)}$ then follows by writing $f^{(k)}/f$ in the form $\Psi_k[f'/f]$. Variants and generalisations of the Tumura-Clunie method, allowing functions g with unrestricted poles, appear in a large number of papers including notably [125, 151, 164].

It was conjectured by Hayman in [71] that if f is meromorphic in the plane and $ff^{(k)}$ has no zeros, for some $k \geq 2$, then $f(z) = \exp(Az + B)$ or $f(z) = (Az + B)^{-m}$, with A, B constants and $m \in \mathbf{N}$. This was proved by Frank [52] for $k \geq 3$ and by Langley [103] for $k = 2$ (see also [9, 105]), while simple examples show that no such result holds for $k = 1$. Generalisations include replacing $ff^{(k)}$ by fF, where F is given by (2) with rational coefficients [15, 54, 104, 148], and by $ff'' - \alpha f'^2, \alpha \in \mathbf{C}$ [8, 105, 123]. Closely linked to Milloux' inequality and Hayman's alternative is the question of whether $G = f^n f'$ must take every finite non-zero value, when f is non-constant meromorphic and $n \in \mathbf{N}$, the connection being that $(n+1)G$ is the derivative of f^{n+1}, which has only multiple zeros and poles. Hayman proved in [71] that this is the case for $n \geq 2$ and f entire, and for $n \geq 3$ when f has poles. Following incremental results by a number of authors [22, 81, 124], the definitive theorem in this direction was proved in [11]: if f is a transcendental meromorphic function in the plane and $m > k \geq 1$ then $(f^m)^{(k)} - a,\ a \neq 0$ has infinitely many zeros. Here the result is proved first for finite order, and the infinite order case is then deduced by applying a renormalisation method from normal families [132, 133, 163]. The closely related question of whether $f' + f^n$ may omit a finite value, which in turn is related to the Tumura-Clunie method, is resolved in [11, 71, 124, 146]. A more recent conjecture [156] asserts that if f is a transcendental meromorphic function then $ff^{(k)}$ takes every finite, non-zero value infinitely often: this is known to be true for $k = 1$ [11], and for $k = 2$ and f entire [107].

A further development from [71] leads to two conjectures which remain open. Hayman observed in [71] that since the derivative f' of a transcendental meromorphic function f has only multiple poles, it follows that f' has at most one finite Picard value. It was subsequently conjectured by Mues [122] that the Nevanlinna deficiencies of $f^{(m)}$ satisfy, for $m \geq 1$,

$$\sum_{a \in \mathbf{C}} \delta(a, f^{(m)}) \leq 1. \tag{3}$$

This was proved by Mues [122] for $m \geq 2$, provided all poles of f are simple. In the general case the best known upper bound for the sum in (3) appears to be 4/3 [84, 161].

The Mues conjecture (3) would follow from a positive resolution of the *Gol'dberg conjecture* that, for a transcendental meromorphic function f and $k \geq 2$,

$$\overline{N}(r, f) \leq N(r, 1/f^{(k)}) + o(T(r, f)) \tag{4}$$

as $r \to \infty$, possibly outside an exceptional set. This is in turn linked to a classical result of Pólya [135] that if f has at least two distinct poles then $f^{(k)}$ has at least one zero, for all sufficiently large k. When f has poles of multiplicity at most $k - 1$ the inequality (4) follows from a lemma of Frank and Weissenborn [55] (see also [154]), so that in particular if f has only simple poles then (3) is true for every positive m. A related inequality was proved in [53, 149]: if $F = L[f]$ as in (2), where the a_j are small functions compared to f, then either

$$N(r, F) \leq N(r, 1/F) + 2N(r, f) + o(T(r, f))$$

outside a set of finite measure or f is a rational function in solutions of the homogeneous equation $L[w] = 0$. This method is connected to Steinmetz' proof [147] of the second main theorem for small functions discussed below. No further results in the direction of (4) appear to be known, although it was proved by Langley in [106] that if f is meromorphic of finite order in the plane and $f^{(k)}$ has finitely many zeros, for some $k \geq 2$, then f has finitely many poles.

Examples abound of meromorphic functions with infinitely many poles such that the first derivative has no zeros, but it was proved in [50] (see also [11]) that if f is transcendental and meromorphic with $\limsup_{r\to\infty} T(r,f)/r = 0$ then f' has infinitely many zeros: the corresponding result with lim sup replaced by lim inf may be found in [83].

We remark that the above results have all been stated for functions meromorphic in the plane. Those which are proved only using properties of the Nevanlinna characteristic admit in some cases generalisation to functions of sufficiently rapid growth in a disk [71] or a half-plane [110]. Some related results for functions of slower growth in the disc appear in [143, 145].

An old conjecture of Nevanlinna was that one can replace constants a_k in the second main theorem by meromorphic functions $a_k(z)$ with the property $T(r,a_k) = o(T(r,f))$. Such functions a_k are usually called "small targets". For the case of an entire function f such a generalization was obtained by Chuang Chi Tai in 1964. Much later, the second main theorem without the ramification term, was proved for meromorphic functions by Osgood, who used methods from number theory [129, 130, 131]. A substantial simplification of Osgood's proof was made by Steinmetz [147], who also used a beautiful idea of Frank and Weissenborn [55]. Osgood and Steinmetz proved that

$$\sum_{k=1}^{q} N(r,(f-a_k)^{-1}) \geq (q-2+o(1))T(r,f),$$

outside of the usual exceptional set. The proof in [147] is simple and elegant; and uses only manipulations with Wronski determinants and the classical lemma on the logarithmic derivative. This makes it suitable for generalizations to holomorphic curves [130, 131, 137]. However, this version of the second main theorem does not take ramification into account. Simple examples like $f(z) = e^z + z$ where $\delta(\infty,f) = \delta(z,f) = 1$ and $N_1(r,f) \sim T(r,f)$ show that one cannot include the term N_1. However, the following form of the second main theorem holds with small targets:

$$\sum_{k=1}^{q} \overline{N}(r,(f-a_k)^{-1}) \geq (q-2+o(1))T(r,f),$$

where $\overline{N}(r,(f-a_k)^{-1})$ is the usual function counting zeros of $f-a$ disregarding multiplicity. This result was recently obtained by Yamanoi [160]. In

[159] he separately treats the case of rational functions a_k when the proof is technically simpler. Yamanoi's proof is very complicated, and it will be hard to generalize to holomorphic curves. Surprisingly, it uses Ahlfors's theory of covering surfaces (and also algebraic geometry, moduli spaces of curves, and combinatorics). The idea to bring Ahlfors's theory to this context has its origin in the work of Sauer [139] who obtained a partial result for rational small targets. One application of Yamanoi's generalization of the second main theorem is the following. *Suppose that f and g are meromorphic functions in* $\mathbf{C}$ *satisfying a relation of the form (1). If the genus of the curve $F(u, v) = 0$ is greater than* 1*, then $T(r, f) + T(r, g) = O(\phi(r))$.* This was conjectured by Eremenko in 1982, and the important special case that $\phi(r) = \ln r$, that is F is a polynomial in all three variables, was proved by Zaidenberg in 1990.

Now we turn to the classic setting. The estimate in the lemma on the logarithmic derivative was improved by Gol'dberg and Grinstein [66]:

$$m(r, f'/f) \leq \ln^{+}\{T(\rho, f)(1 - (r/\rho))^{-1}\} + \text{const},$$

where the constant depends on f. Vojta's analogy (see comments to Chapter I) stimulated new interest in refined estimates for the logarithmic derivative, as well as for the error term

$$S(r, f) = \sum_{j=1}^{q} m(r, a_j, f) + N_1(r, f) - 2T(r, f)$$

in the second main theorem. Miles [118] derived from Gol'dberg's and Grinstein's estimate the following. *Let ψ be a continuous non-decreasing function such that*

$$\int_1^{\infty} \frac{dt}{t\psi(t)} < \infty.$$

Then for every meromorphic function f we have

$$m(r, f'/f) \leq \ln^{+} \psi(T(r, f)) + O(1),$$

outside an exceptional set of finite logarithmic measure. The strongest results on the error term $S(r, f)$ belong to Hinkkanen [85], for example:

$$S(r, f) \leq \ln^{+} \psi(T(r, f)) + O(1),$$

outside an exceptional set of finite logarithmic measure, where ϕ is as before. If one replaces ψ by $t\psi(t)$, then both results will hold outside an exceptional set of finite measure. All presently known results on the error terms in one-dimensional Nevanlinna theory are collected in the book [18].

No analog of the second main theorem holds without an exceptional set of r's. This can be seen from the result of Hayman [72]: *Let $\{E_k\}$ be closed sets of zero logarithmic capacity, and ϕ and ψ arbitrary unbounded increasing functions. Then there exist an entire function f and a sequence $r_k \to \infty$ such that $N(r_k, a, f) \leq \phi(r_k) \ln r_k$ for all $a \in E_k$, while $T(r_k, f) > \psi(r_k)$.*

Chapter IV.

The result of Hayman we just cited shows that the set $E_V(f)$ of Valiron exceptional values, which always has zero capacity, can contain any F_σ set of zero capacity, but a complete description of possible sets $E_V(f)$ is not known.

For meromorphic functions f of finite order, Hyllengren [96] obtained a very precise description of the sets $E_V(f)$. Let us say that a set E satisfies the H-condition if there exist a sequence (a_n) of complex numbers and $\eta > 0$ such that every point of E belongs to infinitely many discs $\{w : |w - a_n| < \exp(-\exp(\eta n))\}$. *For every meromorphic function f of finite order, and every $x \in (0, 1)$, the set $E_V(x, f)$ of those $a \in \mathbf{C}$ for which $\Delta(a, f) > x$ satisfies the H-condition, and vise versa, for every set E satisfying the H-condition, there exist an entire function f and a number $x \in (0, 1)$ such that $\Delta(a, f) > x$, for all $a \in E$.* Notice that the H-condition is much stronger than the condition of zero capacity.

The first example of an entire function of finite order whose deficiency depends on the choice of the origin was constructed by Miles [116]. The order of this function was very large. Then Gol'dberg, Eremenko and Sodin [65] constructed such examples with any given order greater than 5. (For entire functions of order less than 3/2, deficiencies are independent of the choice of the origin.)

Chapter V.

Put $L(r, f) = \ln \mu(r, f)/\ln M(r, f)$, where $\mu(r, f) = \min\{|f(z) : |z| = r\}$. Corollary 3 on p. 232 says that $\limsup_{r\to\infty} L(r, f) \geq -[2\lambda]$. Hayman [70] showed that the same holds with $-2[\lambda]$ replaced by $-2.19 \ln \rho$ when ρ is large enough. For functions of infinite order, he proved

$$\limsup_{r\to\infty} \frac{L(r, f)}{\ln\ln\ln M(r, f)} \geq -2.19.$$

He also constructed examples of entire functions of large finite order for which $\limsup_{r\to\infty} L(r,f) < -1$. Then Fryntov [56], answering a question of Hayman, constructed entire functions of any given order $\rho > 1$ with the same property. Drasin [29] constructed entire functions of order one, maximal type, with the property $M(r,f)\mu(r,f) \to 0$. This may be contrasted with a remarkable theorem of Hayman [73] which says that *if f is an entire function of order one and normal type, and $M(r,f)\mu(r,f)$ is bounded, then $f(z) = c\exp(az)$ for some constants c and a.*

Thus for an entire function of order at least one, $\mu(r,f)$ can decrease at a higher rate than that of increase of $M(r,f)$. The situation changes dramatically if we consider the rate of decrease of $|f(z)|$ on an unbounded *connected* set. Hayman and Kjellberg [78] proved that *for every entire function f and every $K > 1$ all components of the set $\{z : \ln|f(z)| + K\ln M(|z|,f) < 0\}$ are bounded.*

Theorem 1.3′ has been the subject of many deep generalizations. First we mention the famous "spread relation" of Baernstein [2] [4] conjectured by Edrei in [34]: *If f is a meromorphic function of lower order λ, then for every $\epsilon > 0$ there are arbitrary large values of r such that the set of arguments θ where $|f(re^{i\theta})| > 1$ has measure at least*

$$\min\left\{\frac{4}{\lambda}\arcsin\sqrt{\frac{\delta(\infty,f)}{2}},\ 2\pi\right\} - \epsilon. \tag{5}$$

Similar sharp estimates of the measure of the set where

$$\ln|f(re^{i\theta})| > \alpha T(r,f)$$

were given in [1].

Fryntov, Rossi and Weitsman [57, 58] proved that under the assumptions of the spread conjecture, the set $|f(re^{i\theta})| > 1$ must contain an arc of length (5). See also [3] for the sharp lower estimate of the length of the arcs in the set $\{\theta : \ln|f(re^{i\theta}| > \alpha\ln M(r,f)\}$.

Extremal functions for the spread relation and its generalizations were studied extensively, [5, 36, 37, 142].

The new methods introduced by Baernstein [2], [4] are based on the use of subharmonic functions, and especially, on a new type of maximal function, the so-called "star-function", which turned out to be very useful in solving a wide variety of extremal problems of function theory. An account of Baernstein's star function and its main applications is contained in the monograph [79].

One important application of the spread relation is the sharp estimate of the sum of deficiencies of a meromorphic function of lower order $\lambda \leq 1$ [35]. *If a meromorphic function f of lower order λ has at least two deficient values, then*

$$\sum_{a \in \overline{\mathbf{C}}} \delta(a, f) \leq \left\{ \begin{array}{ll} 1 - \cos \pi\lambda, & 0 < \lambda \leq 1/2 \\ 2 - \sin \pi\lambda, & 1/2 < \lambda \leq 1. \end{array} \right. ,$$

The sharp estimate of the sum of deficiencies of a meromorphic function in terms of its order or lower order λ is still not established for $\lambda > 1$. The conjectured extremal functions are described in [32].

The results of §2 show that neither Nevanlinna nor Borel exceptional values need be asymptotic values. On the other hand, Picard exceptional values are asymptotic. A natural question arises, whether any condition of smallness of $N(r, a, f)$ in comparison with $T(r, f)$ will imply that a is an asymptotic value. The basic result belongs to the intersection of the papers [16], [74], and [38]. *Let f be a meromorphic function of lower order $\lambda \leq \infty$. If the order of $N(r, a, f)$ is strictly less than* $\min\{1/2, \lambda\}$ *then a is an asymptotic value.* Example 3 on p. 249 shows that this condition is sharp, if only the lower order of f is taken into account. In [38], a weaker sufficient condition for a to be an asymptotic value is given, that uses both the order and lower order of f. Hayman [74] gives the following refined condition: *if*

$$T(r, f) - \frac{1}{2} r^{1/2} \int_r^\infty t^{-3/2} N(t, a, f) dt \to \infty,$$

then a is an asymptotic value.

The problem on p. 285 of optimal estimation of $\kappa(f)$ for functions of lower order greater than 1 is still open, even for entire functions. It has been solved only for entire functions with zeros on a ray [94].

The best estimates known at this time for entire and meromorphic functions with fixed $\lambda > 1$ are contained in [119, 120]. They are derived from the following sharp inequality which is obtained by the Fourier method:

$$\limsup \frac{N(r, 0) + N(r, \infty)}{m_2(r, f)} \geq \sup_{\lambda^* \leq \lambda \leq \rho^*} \sqrt{2} \frac{|\sin \pi\lambda|}{\pi\lambda} \left\{ 1 + \frac{\sin 2\pi\lambda}{2\pi\lambda} \right\}^{-1/2},$$

where m_2 is the L^2-norm,

$$m_2^2(r, f) = \frac{1}{2\pi} \int_0^{2\pi} (\ln |f(re^{i\theta})|)^2 d\theta,$$

and λ^* and ρ^* are the order and lower order in the sense of Pólya, [31] (see also comments to Chapter II).

In 1929, F. Nevanlinna [127] found that meromorphic functions of finite order satisfying $N_1(r, f) \equiv 0$ have the following properties:

a) 2ρ is an integer, $2\rho \geq 2$,

b) all deficient values are asymptotic, and

c) all deficiencies are rational numbers with denominators at most 2ρ, and their sum equals 2.

It was natural to conjecture that one of the conditions

$$N_1(r, f) = o(T(r, f)), \tag{6}$$

or

$$\sum_{a \in \overline{\mathbf{C}}} \delta(a, f) = 2 \tag{7}$$

implies the properties a), b) and c). Notice, that by the second main theorem, (7) implies (6) for functions of finite order.

It turns out that a strong form of this conjecture holds:

Small Ramification Theorem. *If f is a meromorphic function of finite lower order with the property* (6) *then a), b) and c) hold, and:*

$$T(r, f) = r^{\rho}\ell(r), \tag{8}$$

where ℓ is a slowly varying function in the sense of Karamata.

As a corollary we obtain that conditions (6) and (7) for functions of finite lower order are equivalent.

This result has a long history which begins with theorems of Pfluger and Edrei and Fuchs establishing the case of entire functions (see Corollary 2 on p. 315). Weitsman [158] proved that (7) implies that the number of deficient values is at most 2ρ. Then Drasin [27], [28] proved that for functions of finite order (7) implies a), b) and c) and the regularity condition (8). Eremenko proposed a new potential-theoretic method (see, for example, [48]) which led to a simpler proof of Drasin's theorem. The small ramification theorem in its present form stated above is proved in [44].

These results show that besides the defect relation, there is an additional restriction on defects of functions of finite order: (7) implies that the number of deficient values is finite and all defects are rational.

There are other restrictions as well. Weitsman's theorem [157] says that for functions f of finite lower order

$$\sum_{a\in\overline{\mathbf{C}}} \delta(a,f)^{1/3} < \infty. \tag{9}$$

The story of this theorem is described in Comments to Chapter VII (see p. 576). Weitsman's proof can actually be modified to produce an upper bound depending only on the lower order of f.

The second restriction concerns functions of finite order having a defect equal to 1. Lewis and Wu proved that such functions satisfy

$$\sum_{a\in\overline{\mathbf{C}}} \delta(a,f)^{1/3-\alpha} < \infty,$$

with some absolute constant $\alpha > 0$. Their proof gives $\alpha = 2^{-264}$, which is far from what is expected. (Lewis and Wu state their result for entire functions but their proof applies to all functions with $\delta(a,f) = 1$ for some a.)

Examples of entire functions of finite order with the property $\delta(a_n,f) \geq c^n$ for some $c \in (0,1)$ are constructed in [45], but a large gap remains between these examples and the result of Lewis and Wu.

Recent research on value distribution meromorphic functions of the form

$$\sum \frac{c_k}{z - z_k}, \quad \sum \frac{|c_k|}{|z_k|} < \infty \tag{10}$$

was mainly concentrated on the functions with $c_k > 0$. Such functions are (complex conjugate to) gradients of subharmonic functions of genus zero with discrete mass. The main conjecture is that every function of the form (10) has zeros. This was proved in [50] under the additional assumption that $\inf c_k > 0$.

Chapter VI

Entire functions whose zeros lie on (or are close to) finitely many rays were intensively studied. Under certain conditions, one can estimate $\delta(0,f)$ from below, as in Corollary 4 on p. 350. The strongest results in this direction belong to Hellerstein and Shea [91] and Miles [115]. One of the results of [91] says that $\delta(0,f) > B_q(\theta_1,...,\theta_n)$ *for all entire functions of genus* q *with zeros on the rays* $\arg z \in \{\theta_1,...,\theta_n\}$, *and* $B_q \to 1$ *when* $q \to \infty$

while the rays remain fixed. In the case of one ray, they obtained $B_q(\theta) = 1 - (\pi^2 e^{-1} + o(1))/\ln q,\ q \to \infty$. For entire functions of infinite order with zeros on a ray, Miles [115] proved that $N(r,0,f)/T(r,f) \to 0$ as $r \to \infty$ avoiding an exceptional set of zero logarithmic density. However, it may happen that $\delta(0,f) = 0$ for such functions, as Miles shows by an example constructed in the same paper.

Hellerstein and Shea [91] also considered meromorphic functions of finite order whose zeros $\{z_n\}$ and poles $\{w_n\}$ lie in opposite sectors $|\arg z_n| \leq \eta$ and $|\arg w_n - \pi| \leq \eta$, where $0 \leq \eta < \pi/6$. For such functions, they obtained a sharp estimate of $\kappa(f)$ (definition on p. 285) from above.

For entire functions with zeros on finitely many rays, there are relations between the order and lower order (see p. 344). There relations were further investigated in [117, 60] and [136].

Miles [117] considers the class of meromorphic functions f whose zeros belong to a finite union of rays X and poles belong to a finite union of rays Y, where $X \cap Y = \emptyset$, and such that the exponent of convergence of the union of zeros and poles is a given number q. He then produces a non-negative integer $p = p(q, X, Y)$ such that

$$\lim_{r\to\infty} \frac{T(r,f)}{r^p} = \infty \quad \text{if} \quad p > 0 \quad \text{and} \quad \lim_{r\to\infty} \frac{T(r,f)}{\ln r} = \infty \quad \text{if} \quad p = 0,$$

and these growth estimates are sharp in the considered class. The integer p depends in a subtle way on the arithmetical properties of the arguments of the rays X and Y, and this integer is in general hard to compute or estimate.

Gleizer [60] considers entire functions with zeros on n rays. If $n = 1$ or $n = 2$, we have $\rho \leq [\lambda] + n$, where $[\ ]$ is the integer part. This follows from Theorem 1.1, Chapter VI. However, if $n = 3$, then the difference $\rho - \lambda$ can be arbitrarily large. In this case, Gleizer proved that $[\rho] \leq 3([\lambda] + 1)$. For arbitrary n, Qiao [136] proved that $\rho \leq 4^{q-1}([\lambda] + 1)$.

In [59, 61], Gleizer extended Theorem 4.1 by taking into account not only the order but the lower order in the sense of Pólya. He used Baernstein's star-function.

There has been remarkable progress in the problems considered in §5. The conjecture of Pólya and Wiman stated on p. 417 was proved by Hellerstein and Williamson [92, 93]. *If f is a real entire function such that all zeros of $ff'f''$ are real then f belongs to the Laguerre–Pólya class.* In [95] the same authors with Shen classified all entire functions (not necessarily real) with the property that $ff'f''$ has only real zeros. The classification of meromorphic functions with the property that all their derivatives have only real zeros was achieved by Hinkkanen [86, 87, 88, 89, 90].

Sheil-Small [144] proved a conjecture of Wiman (1911), that every real entire function of finite order with the property that ff'' has only real zeros belongs to the Laguerre–Pólya class. Bergweiler, Eremenko and Langley [13] extended Sheil-Small's result to functions of infinite order. Then Langley [109] extended this result to the derivatives of higher orders: *If f is a real entire function of infinite order, with finitely many non-real zeros, then $f^{(k)}$ has infinitely many non-real zeros for every $k \geq 2$.*

For real entire functions of finite order with finitely many non-real zeros, that do not belong to the Laguerre–Pólya class, Bergweiler and Eremenko [12] proved that the number of non-real zeros of $f^{(k)}$ tends to infinity as $k \to \infty$. Together with Langley's result, this confirms another conjecture of Pólya (1943).

Chapter VII

The inverse problem (as stated on p. 487) was completely solved by Drasin [25]. A simplified proof is given in [26]. The general idea is the same as in Chapter VII: quasiconformal surgery and a version of the theorem of Belinskii and Teichmüller are involved. However, unlike in Chapter VII, Drasin does not construct the Riemann surface spread over the sphere explicitly but uses a more flexible technique.

Theorem 8.1 in Chapter VII actually gives a complete solution of the inverse problem for finitely many deficient values in the class of meromorphic functions of finite order. (This was not known in 1970 when the book was written. That condition 3 of this Theorem 8.1 is necessary follows from the small ramification theorem above).

On the narrow inverse problem in the class of meromorphic functions of finite order with infinitely many deficiencies, there is the following result [42]:

Let $\{a\}$ be an arbitrary infinite countable subset of $\overline{\mathbf{C}}$, and $\{\delta_n\}$ positive numbers satisfying the following conditions:
(i) $\delta_n < 1$,
(ii) $\sum_n \delta_n < 2$, *and*
(iii) $\sum_n \delta_n^{1/3} < \infty$.
Then there exists a meromorphic function f of (large) finite order such

that $\delta(a_n, f) = \delta_n$, *and* f *has no other deficient values.*

The order of this function depends on the quantities in the right hand sides of (i), (ii) and (iii).

Conditions (ii) and (iii) are necessary because of the small ramification theorem, and Weitsman's theorem (see comments to Chapter V above). Condition (i) cannot be removed because of the Lewis and Wu theorem stated above, but it is not known what the precise condition on δ_n is, if $\delta_1 = 1$.

The class of meromorphic functions with finitely many critical and asymptotic values which was used in Chapter VII to investigate the inverse problem is interesting independently of this application. Let us call this class S. The first general result on functions of this class belongs to Teichmüller, who proved that the second main theorem becomes an asymptotic equality for functions of this class:

$$\sum_{j=1}^{q} m(r, a_j, f) + N_1(r, f) = 2T(r, f) + Q(r, f),$$

where a_j are all critical and asymptotic values.

Langley [108] found that the growth of a function $f \in S$ cannot be arbitrary:

$$c(f) := \liminf_{r\to\infty} \frac{T(r, f)}{\ln^2 r} > 0.$$

This constant $c(f)$ can be arbitrarily small, but in the case that f has only three critical and asymptotic values, we have [49] $c(f) \geq \sqrt{3}/(2\pi)$ and this is best possible. On the other hand, there are no restrictions from above on the growth of functions of class S [113].

Class S plays an important role in holomorphic dynamics (iteration of entire and meromorphic functions), see, for example, the survey [7]. In [23] an application of almost periodic ends is given. In [32] the method of Chapter VII is extended to a new class of Riemann surfaces which the authors call "Lindelöfian ends". The corresponding functions have infinitely many critical values and thus do not belong to the class S.

Appendix.

Govorov's original proof of the Paley conjecture was a byproduct of his research on the Riemann Boundary problem with infinite index [68]. His theorem was generalized to meromorphic functions by Petrenko, to subharmonic functions in $\mathbf{R}^n$ by Dahlberg, and to entire functions of several complex variables by Khabubullin, see his survey [98].

Petrenko introduced the following quantity which he called the "deviation of f from the point a".

$$\beta(a,f)=\liminf_{r\to\infty}\frac{\ln^+ M(r,(f-a)^{-1})}{T(r,f)}. \tag{11}$$

This differs from the defect in one respect: the uniform norm of $\ln^+|f(re^{i\theta})-a|^{-1}$ stands in the numerator instead of the L^1 norm. Petrenko's generalization of Govorov's theorem proved in the Appendix can be restated as:

$$\beta(a,f)\leq\pi\lambda \tag{12}$$

for all meromorphic functions of lower order λ and all $a\in\overline{\mathbf{C}}$. This was the starting point of a study of deviations $\beta(a,f)$ by Petrenko and others. The results obtained before 1978 are summarized in his book [134]. The main difference between the theory of deviations and the theory of defects is the absence of a first main theorem: there is no simple relation between $\beta(a,f)$ and solutions of the equation $f(z)=a$.

We only present a sample of the results. By analogy with defects, one can expect that the set of exceptional values in the sense of Petrenko

$$P(f):=\{a\in\overline{\mathbf{C}}:\beta(a,f)>0\}$$

is small. This is indeed the case: *for every meromorphic function f, the set $P(f)$ has zero logarithmic capacity; for functions of finite lower order it is at most countable (but may have the power of continuum for functions of infinite lower order).* The following analog of the Defect relation for functions of finite lower order was established by Marchenko and Shcherba [112]:

$$\sum_{a\in\overline{\mathbf{C}}}\beta(a,f)\leq\begin{cases}2\pi\lambda, & \lambda\geq 1/2,\\ 2\pi\lambda\csc\pi\lambda, & \lambda<1/2.\end{cases}$$

Moreover, an analog of Weitsman's theorem (see comments to Chapter V) holds: *for functions f of finite lower order we have*

$$\sum_{a\in\overline{\mathbf{C}}}\beta(a,f)^{1/2}<\infty,$$

and the exponent 1/2 is best possible. A version Baernstein's spread relation also holds with deviations instead of deficiencies [134]. It is worth mentioning here that, according to Baernstein [4], the idea of introducing the star function that led to the proof of the spread relation occurred under the influence of Petrenko's proof of (12).

The inverse problem for deviations turned out to be simpler than the inverse problem for deficiencies. A complete solution for functions of finite order is given in [42]: *For every at most countable set* $\{a_n\}$ *of points and every sequence of positive numbers* β_n *satisfying the condition* $\sum \beta_n^{1/2} < \infty$, *there exists a meromorphic function* f *of finite order such that* $\beta(a_n, f) = \beta_n$ *and* $\beta(a, f) = 0$ *for* $a \notin \{a_n\}$.

In general, there is no relation between the sets $E_N(f)$ and $P(f)$: *for every pair* (A, B) *of at most countable subsets of* $\overline{\mathbf{C}}$, *there exists a meromorphic function* f *of any given non-zero order such that* $E_N(f) = A$ *and* $P(f) = B$ [64, 65]. On the other hand, if $T(2r, f) = O(T(r, f))$ then $P(f) = E_V(f)$ [41].

Bergweiler and Bock [10] found an analog of (12) for functions of infinite lower order. The idea was to replace $T(r, f)$ in the denominator of (11) by $A(r, f)$. Notice that if one uses the Ahlfors definition of $T(r, f)$ then $A(r, f) = dT(r, f)/d\ln r$, for example, if $T(r, f) = r^\lambda$ then $A(r, f) = \lambda T(r, f)$. Bergweiler and Bock proved that *for every meromorphic function* f *of order at least* 1/2 *and every* $a \in \overline{\mathbf{C}}$ *we have*

$$b(a, f) := \liminf_{r\to\infty} \frac{\ln^+ M(r, (f-a)^{-1})}{A(r, f)} \leq \pi,$$

and then Eremenko [46] established the following analog of the Defect Relation:

$$\sum_{a\in\overline{\mathbf{C}}} b(a, f) \leq \begin{cases} 2\pi, & \lambda > 1/2, \\ 2\pi \sin \pi\lambda, & \lambda \leq 1/2 \end{cases},$$

assuming that there are at least two values a with $b(a, f) > 0$. It follows that for every meromorphic function the set $\{a \in \overline{\mathbf{C}} : b(a, f) > 0\}$ is at most countable.

Even Drasin's theorem on the extremal functions for the defect relation (see Comments to Chapter V) has its analog for $b(a, f)$ [47]:

If f *is of finite lower order and*

$$\sum_{a\in\overline{\mathbf{C}}} b(a, f) = 2\pi$$

then the following limit exists

$$\lim_{r\to\infty} \frac{\ln T(r,f)}{\ln r} = \frac{n}{2},$$

where n *is an integer, and* $b(a,f) = \pi/n$ *or* 0 *for every* $a \in \overline{\mathbf{C}}$.

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