SPECTRAL INCLUSION AND ANALYTIC CONTINUATION

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Introduction

Let a be an element in a complex Banach algebra with unit, and let r be a nonnegative number. The Gelfand spectral radius formula implies that the spectrum of a is included in the disk $\{z \in \mathbf{C} : |z| \leqslant r\}$ if and only if

$$\lim_{n\to\infty} \|a^n\|^{1/n} \leqslant r.$$

PROBLEM 1. Is there a similar criterion for spectral inclusion into other compact sets?

In this paper we prove that for every compact set $K \subset \mathbf{C}$ with connected complement, there exists a sequence $\{P_n\}_{n=0}^{\infty}$ of monic polynomials with $\deg P_n = n$, and a nonnegative number r, such that the spectrum of a is included in K if and only if

$$\limsup_{n\to\infty} \|P_n(a)\|^{1/n} \leqslant r. \tag{1}$$

As shown in Section 2, the condition that the complement of K is connected is also necessary for existence of such a sequence of polynomials.

We also obtain a solution to the following more general problem.

PROBLEM 2. Given an analytic germ $f(z) = \sum_{k=0}^{\infty} f_k z^{-k-1}$ at ∞ with values in a Banach space, and a domain $\Omega \in \bar{\mathbf{C}}$ such that $\infty \in \Omega$, $\Omega \neq \bar{\mathbf{C}}$, determine whether f has analytic continuation to Ω.

By this we mean that f can be analytically continued along every path in Ω starting at ∞.

We prove that for every such domain Ω there exists a sequence of monic polynomials $P_n(z) = \sum_{j=0}^{n} p_j^{(n)} z^j$, $\deg P_n = n$, and a number $r \geqslant 0$, such that f has analytic continuation into Ω if and only if

$$\limsup_{n\to\infty} \left\| \sum_{j=0}^{n} p_j^{(n)} f_j \right\|^{1/n} \leqslant r. \tag{2}$$

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For Ω simply connected and f complex-valued, this result was proved by Havin [**3**, **4**] with $\{P_n\}_{n=0}^{\infty}$ being the sequence of Faber polynomials associated with $K = \bar{\mathbf{C}} \setminus \Omega$, and r the logarithmic capacity of K.

In Section 1 we state and prove our main results. Since the construction of the polynomials which yield the solutions to these problems depends on the uniformization theorem, it is desirable to find more explicit constructions. This issue is discussed in Section 2, where we give simple conditions for a sequence of polynomials to determine spectral inclusion.

Throughout this paper, K will denote a non-empty compact set in $\mathbf{C}$ with connected complement, and the domain $\bar{\mathbf{C}} \setminus K$ will be denoted by Ω. For $r \geqslant 0$, we shall denote $\Delta_r = \{z \in \bar{\mathbf{C}} : |z| > r\}$. In what follows, A will denote a complex Banach algebra with unit e, and the spectrum of an element a in A will be denoted by $\sigma(a)$.

1. *Main results*

Let ψ be a formal Laurent series over $\mathbf{C}$ of the form

$$\psi = z + \sum_{j=0}^{\infty} \psi_j z^{-j}. \tag{3}$$

A straightforward computation shows that for every $w \in \mathbf{C}$,

$$(\psi - w)^{-1} = \sum_{n=0}^{\infty} P_n(w) z^{-n-1}, \tag{4}$$

where the sequence $P_n(w)$ is determined recursively by the equations $P_0(w) = 1$, $P_1(w) = w - \psi_0$ and

$$P_n(w) = (w - \psi_0)P_{n-1}(w) - \sum_{j=0}^{n-2} \psi_{n-j-1} P_j(w), \quad n \geqslant 2.$$

These equations show that P_n is a monic polynomial of degree n. The sequence $\{P_n\}_{n=0}^{\infty}$ will be called, in the sequel, the *standard sequence of polynomials associated with* ψ.

The compositional inverse φ of ψ is a formal Laurent series of the form

$$\varphi = w + \sum_{j=0}^{\infty} \varphi_j w^{-j}. \tag{5}$$

One can show that for $n = 0, 1, \ldots,$

$$(n+1)^{-1} \left(\varphi^{n+1}\right)' = P_n(w) + O(w^{-2}), \tag{6}$$

where $O(w^{-2})$ stands for a formal Laurent series of the form $\sum_{j=2}^{\infty} b_j w^{-j}$.

Equation (6) yields an alternative definition of the sequence of standard polynomials associated with ψ. We recall that the Faber polynomials F_n associated with ψ [**5**, Section 18] are defined by the equation

$$\psi' (\psi - w)^{-1} = \sum_{n=0}^{\infty} F_n(w) z^{-n-1}, \tag{7}$$

or alternatively by the equation

$$\varphi^n(w) = F_n(w) + O(w^{-1}), \quad n = 0, 1, \ldots. \tag{8}$$

It follows from (6) and (8) that

$$F_n' = nP_{n-1}, \quad n = 0, 1, \ldots. \tag{9}$$

By the uniformization theorem (a simple proof for the special case of domains in $\bar{\mathbf{C}}$ is given in [**1**, Chapter VI, §1]), there exists a unique number $r(K) \geqslant 0$, and a unique normalized universal covering $\psi_K : \Delta_{r(K)} \to \Omega$ which has a Laurent expansion of the form (3) at infinity. The sequence of standard polynomials associated with this Laurent series will be denoted by $\{P_n^K\}_{n=0}^{\infty}$, and the corresponding sequence of Faber polynomials will be denoted by $\{F_n^K\}_{n=0}^{\infty}$.

The constant $r(K)$ vanishes if and only if K consists of one or two points. If Ω is simply connected, that is, K is connected, then $r(K)$ coincides with the logarithmic capacity $c(K)$. If K is disconnected and contains more than two points, then $r(K) > c(K)$.

Examples. (i) If K is a disk of radius r centred at z_0, then $\psi_K(z) = z + z_0$, $r(K) = r$ and

$$P_n^K(z) = F_n^K(z) = (z - z_0)^n.$$

(ii) If $K = [-1, 1]$, then $\psi_K(z) = z + 1/(4z)$, $r(K) = 1/2$,

$$P_n^K(\cos t) = \frac{\sin(n+1)t}{2^n \sin t}$$

and

$$F_n^K(\cos t) = 2^{-n+1} \cos nt.$$

These are the monic Chebyshev polynomials of the second and the first kind, respectively.

(iii) If $K = \{-i, i\}$, then $\psi_K(z) = \cot(1/z)$, $r(K) = 0$,

$$P_n^K(\cot w) = \frac{(-1)^n}{(n+1)!\,(1+\cot^2 w)} \frac{d^{n+1}}{dw^{n+1}}(\cot w)$$

and

$$F_n^K(\cot w) = \frac{(-1)^{n-1}}{(n-1)!} \frac{d^{n-1}}{dw^{n-1}}(\cot w).$$

Our main results are as follows.

Theorem 1. *If $a \in A$, then $\sigma(a) \subset K$ if and only if* (1) *holds with $P_n = P_n^K$ and $r = r(K)$.*

Theorem 2. *Let $f(z) = \sum_{k=0}^{\infty} f_k z^{-k-1}$ be an analytic germ at ∞ with values in a Banach space. Then f has analytic continuation to Ω if and only if* (2) *holds with $P_n = P_n^K$ and $r = r(K)$.*

Taking f in Theorem 2 to be the resolvent of an element $a \in A$, we obtain Theorem 1. However, we start with an independent proof which applies also in a more general case (see Remark (i) below).

Proof of Theorem 1. Let $r_0 > r$ be such that $|\psi_K(z)| > \|a\|$, for $|z| > r_0$. Since the resolvent $z \mapsto R_a(z) = (ze - a)^{-1}$ is a holomorphic A-valued function in $\Delta_{\|a\|}$ which vanishes at infinity, the function $F = R_a \circ \psi_K$ is a holomorphic A-valued function which vanishes at infinity, and by (4), its Laurent expansion in Δ_{r_0} is given by

$$F(z) = \sum_{n=0}^{\infty} P_n^K(a) z^{-n-1}, \quad z \in \Delta_{r_0}. \tag{10}$$

If $\sigma(a) \subset K$, then $\mathscr{R}_a$ is holomorphic in Ω, and therefore $F(z)$ is holomorphic in $\Delta_{r(K)}$. Hence the series in (10) converges also in that domain. Thus, by the Cauchy–Hadamard theorem, we obtain that (1) is satisfied.

Conversely, assume that condition (1) holds. Then it follows from (10) that F has a holomorphic extension F_1 onto $\Delta_{r(K)}$. By the permanence principle, we obtain from (10) that

$$(\psi_K(z)e - a)F_1(z) = F_1(z)(\psi_K(z)e - a) = e$$

holds for every z in the domain $\Delta' = \{z \in \Delta_{r(K)} : \psi_K(z) \neq \infty\}$. This implies that

$$\mathbf{C} \setminus K = \psi_K(\Delta') \subset \mathbf{C} \setminus \sigma(a),$$

and the proof is complete.

Remarks. (i) It follows from the proof given above that if ψ is a meromorphic function in Δ_r, $r \geqslant 0$, which maps Δ_r onto Ω, and which has Laurent expansion near infinity of the form (3), and if $\{P_n\}_{n=0}^{\infty}$ is the standard sequence of polynomials associated with ψ, then $\sigma(a) \subset K$ if and only if (1) holds.

(ii) If Ω is the unbounded component of the resolvent set of a, then (10) gives the Taylor series expansion of the resolvent on the universal covering of Ω. This might be useful in spectral theory.

(iii) For $K = [-1, 1]$,

$$\limsup_{n\to\infty} |P_n^K(0)|^{1/n} = 1/2 \quad \text{and} \quad \liminf_{n\to\infty} |P_n^K(0)|^{1/n} = 0.$$

This shows that the upper limit in Theorem 1 cannot be replaced in general by a limit, even for the zero element in a Banach algebra.

Proof of Theorem 2. It follows from the monodromy theorem that f has analytic continuation to Ω if and only if the function $g = f \circ \psi_K$ has holomorphic extension to $\Delta_{r(K)}$. Assume that the Laurent expansion of g at infinity is given by $g(z) = \sum_{n=0}^{\infty} g_n z^{-n-1}$, and denote by u the identity function on $\mathbf{C}$, and by φ_K the compositional inverse of ψ_K. Applying the residue theorem, changing variables and using (6), we obtain that for every nonnegative integer n,

$$\begin{aligned} g_n &= -\mathrm{res}_\infty[u^n g] \\ &= -\mathrm{res}_\infty[u^n (f \circ \psi_K)] \\ &= -\mathrm{res}_\infty[\varphi_K^n \varphi_K' f] \\ &= -(n+1)^{-1} \mathrm{res}_\infty[(\varphi_K^{n+1})' f] \\ &= -\mathrm{res}_\infty[P_n^K f] = \sum_{j=0}^{n} p_j^{(n)} f_j, \end{aligned}$$

and this implies the desired conclusion, by the Cauchy–Hadamard formula.

REMARKS. (i) Havin's theorem mentioned in the introduction is proved in [**7**, Chapter II, §1] along the same lines as our proof of Theorem 2. However, for disconnected K, one cannot replace in Theorems 1 and 2 the polynomials P_n^K by the Faber polynomials F_n^K. To see this, we recall that the Faber polynomials F_n^K are defined by equation (7). If K is disconnected, then ψ_K has infinitely many poles. Let $z_0 \in \Delta_{r(K)}$ be the pole with largest absolute value, $|z_0| = r_0$. Taking in (7) $A = \mathbf{C}$ and $a \in K$, we conclude from the Cauchy–Hadamard formula that

$$\limsup_{n\to\infty} |F_n^K(a)|^{1/n} = r_0 > r(K).$$

On the other hand, $\sigma(a) = \{a\}$, so by Theorem 1,

$$\limsup_{n\to\infty} |P_n^K(a)|^{1/n} \leqslant r(K).$$

(ii) One can show that in Theorem 2 the standard polynomials can be replaced by the polynomials $F_n^K(w) - F_n^K(w_0)$, where w_0 is an arbitrary point in K.

(iii) In Theorem 2, the assumption that the germ f is *analytic* at infinity can be omitted; that is, the criterion can be applied to formal Laurent germs as well. For this, one should apply a formal version of the Bürmann–Lagrange theorem [**5**, Theorem 18c].

2. *Sequences determining spectral inclusion*

We say that a sequence of polynomials $\{P_n\}_{n=0}^{\infty}$ and a number $r \in [0, \infty)$ *determine spectral inclusion* into K if for every $a \in A$, the condition $\sigma(a) \subset K$ is equivalent to condition (1). Similarly, we say that a sequence of polynomials $\{P_n\}_{n=0}^{\infty}$ and a number $r \in [0, \infty)$ *determine analytic continuation* to Ω if every analytic germ f at ∞ with values in a Banach space has analytic continuation to Ω if and only if (2) holds. If a sequence of polynomials and a nonnegative number determine analytic continuation to Ω, then they determine spectral inclusion into $K = \bar{\mathbf{C}} \setminus \Omega$. We shall see in the sequel that the converse is not true.

First, we derive some conditions for a sequence of polynomials $\{P_n\}_{n=0}^{\infty}$ and a number $r \geqslant 0$ to determine spectral inclusion into K. The following conditions are necessary:

$$\limsup_{n\to\infty} |P_n(w)|^{1/n} > r, \quad \text{for } w \in \mathbf{C} \setminus K, \tag{11}$$

and

$$\limsup_{n\to\infty} \max_K |P_n|^{1/n} \leqslant r. \tag{12}$$

The first condition follows from the fact that the spectrum of every element w of the Banach algebra $\mathbf{C}$ coincides with the singleton $\{w\}$, and the second condition follows from the fact that the spectrum of the function $a : z \mapsto z$ in the Banach algebra $C(K)$ of all complex-valued continuous functions with the supremum norm coincides with K.

It also follows from these observations that a necessary condition for the existence of a sequence of polynomials which determines spectral inclusion into K with some number $r \geqslant 0$ is that the complement $\mathbf{C} \setminus K$ is connected. Otherwise, if V is a bounded component of the complement, then by the maximum principle, for every $w \in V$, and every polynomial P, we have that $|P(w)| \leqslant \max_K |P|$, and consequently, conditions (11) and (12) cannot be satisfied simultaneously.

Sufficient conditions for spectral inclusion are given by the following.

THEOREM 3. *If $\{P_n\}_{n=0}^{\infty}$ and r satisfy* (11), *and for every $\rho > r$ there exists a neighbourhood V of K such that*

$$\limsup_{n\to\infty} \sup_V |P_n|^{1/n} \leqslant \rho, \tag{13}$$

then $\{P_n\}_{n=0}^{\infty}$ and r determine spectral inclusion into K.

Proof. Let $a \in A$, and assume that (11) holds and $w \in \sigma(a)$. By the spectral mapping theorem, $P_n(w) \in \sigma(P_n(a))$, $n = 0, 1, 2, \dots$, and therefore, since the spectral radius of an element in a Banach algebra does not exceed its norm, $|P_n(w)| \leqslant \|P_n(\sigma(a))\|$. Thus by (11), $\limsup_{n\to\infty} |P_n(w)|^{1/n} \leqslant r$, and therefore by (12), $w \in K$. This proves that $\sigma(a) \subset K$.

Conversely, assume $\sigma(a) \subset K$. Let $\rho > r$, and consider an open neighbourhood V of K such that (13) holds. Let Γ be a 1-cycle in $V \setminus K$ which has index 1 with respect to every point of K. Then, by the analytic functional calculus for elements in a Banach algebra,

$$P_n(a) = \frac{1}{2\pi i} \int_\Gamma P_n(z) R_a(z)\, dz, \quad n = 0, 1, \dots,$$

whence

$$\|P_n(a)\| \leqslant \left(\frac{1}{2\pi} \int_\Gamma \|R_a(z)\|\, |dz| \right) \sup_V |P_n|, \quad n = 0, 1, \dots.$$

By (13) this implies that $\limsup_{n\to\infty} \|P_n(a)\|^{1/n} \leqslant \rho$, and since this holds for every $\rho > r$, we obtain (11).

REMARKS. (i) One can deduce Theorem 1 also from Theorem 3.

(ii) As a simple application of Theorem 3, we obtain that if a set K is finite, then the sequence of polynomials $Q_n(z) = \prod_{\lambda\in K}(z-\lambda)^n$, $n = 0, 1, \dots$, and $r = 0$, determine spectral inclusion into K. This also follows from the spectral mapping theorem and Gelfand's formula.

EXAMPLE. Here is a sequence of polynomials which determines spectral inclusion into $K = \{z : |z| \leqslant 1\}$ with $r = 1$, but not analytic continuation into $\Delta_1 = \bar{\mathbf{C}} \setminus K$ for any r. Let $P_n(z) = z^n - a_n z^{n-1}$ with $a_n = 2 + 1/n$. Then $|P_n(z)|^{1/n} \to |z|$ for every $z \in \mathbf{C}$, and the convergence is uniform in $\{z : |z| \leqslant 3/2\}$, so conditions (11) and (13) are satisfied for K and $r = 1$. Thus $\{P_n\}_{n=0}^{\infty}$ and $r = 1$ determine spectral inclusion into K by Theorem 3. If we consider the power series $f(z) = \sum_{n=0}^{\infty} c_n z^{-n-1}$ with $c_0 = 1$ and $c_n = c_{n-1} a_n$, then its maximal region of convergence is Δ_2, hence f has no analytic continuation to any region containing the circle $|z| = 2$. But, on the other hand, $\operatorname{res}_\infty [P_n f] = 0$ for all n, so (2) holds for every $r \geqslant 0$.

Now we discuss briefly how to construct sequences of polynomials which satisfy conditions (11) and (13) if $\partial\Omega$ is regular for the Dirichlet problem. Let U^K be the equilibrium potential for K, that is, $U^K(w) = c(K)$, $w \in K$, $U^K(w) > c(K)$, $w \in \Omega$, and U^K is harmonic in $\Omega \setminus \{\infty\}$. A sequence of polynomials $\{P_n\}_{n=0}^{\infty}$ is called *nearly extremal* for K if

$$\lim_{n\to\infty} |P_n(w)|^{1/n} = \exp U^K(w) \quad \text{uniformly on compacta in } \mathbf{C} \setminus K. \tag{14}$$

Using Theorem 3, one can show that every nearly extremal sequence determines spectral inclusion into K with $r = c(K)$.

Examples of nearly extremal sequences are the sequence of Fekete polynomials for K [**6**, Chapter V], and if K is connected, the sequence of classical Faber polynomials [**7**, Chapter II]. Also, any classical system of orthogonal polynomials is nearly extremal for $[-1, 1]$ (see [**8**]).

If K has zero capacity, then the Evans potential [**6**] can be used instead of the equilibrium potential. In this case, $U^K(w) = -\infty$ for $w \in K$ and $U^K(w) > -\infty$ for $w \in \Omega$, so any sequence of polynomials satisfying (14) determines spectral inclusion into K with $r = 0$.

This fact is related to a result of Halmos. In [**2**] he proved that $\sigma(a)$ has zero capacity if and only if a is *quasialgebraic*, that is, there is a sequence of monic polynomials $\{Q_n\}_{n=0}^{\infty}$ for which the sequence $d_n = \deg Q_n$ is increasing, such that $\lim_{n\to\infty} \|Q_n(a)\|^{1/d_n} = 0$. The argument above gives one part of this result: if $\operatorname{cap}(\sigma(a)) = 0$, then a is quasialgebraic. Indeed, if $K = \sigma(a)$ has zero capacity, and the sequence of polynomials $\{P_n\}_{n=0}^{\infty}$ satisfies (14) with Evans' potential U^K, then by Theorem 3, $\lim_{n\to\infty} \|P_n(a)\|^{1/n} \to 0$, and therefore the element a is quasialgebraic.

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