Real entire functions of infinite order and a conjecture of Wiman

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Abstract

We prove that if f is a real entire function of infinite order, then ff'' has infinitely many non-real zeros. In conjunction with the result of Sheil-Small for functions of finite order this implies that if f is a real entire function such that ff'' has only real zeros, then f is in the Laguerre-Pólya class, the closure of the set of real polynomials with real zeros. This result completes a long line of development originating from a conjecture of Wiman of 1911.

1 Introduction

An entire function is called real if it maps the real line into itself. We recall that the Laguerre–Pólya class (LP) consists of entire functions which can be approximated by real polynomials with only real zeros, uniformly on compact subsets of the plane. It is easy to see that LP is closed under differentiation; in particular, all derivatives of a function of the class LP have only real zeros.

Theorem 1.1 *If f is a real entire function and ff'' has only real zeros then f belongs to the class LP.*

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Here one cannot replace ff'' by ff' as the example $f(z) = \exp(\sin z)$ shows. Further, the hypothesis that f is real is essential, because of the example $f(z) = \exp(e^{iz})$ which is due to Edrei [5]. For real entire functions with finitely many zeros, all of them real, Theorem 1.1 was proved in [3].

Theorem 1.1 confirms a conjecture going back to Wiman (1911). Ålander [2, p. 2] seems to state Wiman's conjecture only for functions of finite genus, but in the later statements of the conjecture by Levin and Ostrovskii [18, first footnote on p. 324] and by Hellerstein and Williamson [11, footnote on p. 229], [12] and [4, Probl. 2.64] there is no restriction on the genus. For functions of finite genus, Wiman made a more precise conjecture [1], which was proved by Sheil-Small (Theorem A below).

A weaker conjecture by Pólya [22] that if a real entire function f and all its derivatives have only real zeros then $f \in LP$, was confirmed by Hellerstein and Williamson [11, 12]. They proved that *for a real entire function f the condition that $ff'f''$ has only real zeros implies that $f \in LP$.* Theorem 1.1 shows that one can drop the assumption on the zeros of f' in this result.

For every integer $p \geq 0$ denote by V_{2p} the set of entire functions of the form

$$f(z) = \exp(-az^{2p+2})g(z),$$

where $a \geq 0$ and g is a real entire function with only real zeros of genus at most $2p+1$, and set $U_0 = V_0$ and $U_{2p} = V_{2p} \backslash V_{2p-2}$ for $p \geq 1$. Thus the class of all real entire functions of finite order with real zeros is represented as a union of disjoint subclasses $U_{2p},\ p = 0, 1, \ldots$. According to a theorem of Laguerre [15] and Pólya [21], $LP = U_0$. The following result was conjectured by Wiman [1, 2] and proved by Sheil-Small [24]:

Theorem A (Sheil-Small) *If $f \in U_{2p}$ then f'' has at least $2p$ non-real zeros.*

In particular, if f is a real entire function of finite order and all zeros of ff'' are real then $f \in U_0 = LP$. In a recent paper [6] Edwards and Hellerstein extended Theorem A to real entire functions with finitely many non-real zeros. In particular they proved [6, Corollary 5.2] that *if $f = gh$, where $h \in U_{2p}$ and g is a real polynomial, then $f^{(k)}$ has at least $2p$ non-real zeros, for each $k \geq 2$.*

The main result of this paper can be considered as an extension of Theorem A to functions of infinite order:

Theorem 1.2 *For every real entire function f of infinite order, ff'' has infinitely many non-real zeros.*

Theorem 1.1 is a corollary of Theorem A and Theorem 1.2.

Applying Theorem 1.2 to functions of the form

$$f(z) = \exp \int_0^z g(\zeta)\, d\zeta$$

we obtain

Corollary 1.1 *For every real transcendental entire function g, the function $g' + g^2$ has infinitely many non-real zeros.*

For polynomials g the corresponding result was conjectured in [4, Probl. 2.64 and 4.28] and proved in [24]: *If g is a real polynomial then $g' + g^2$ has at least* $\deg g - 1$ *non-real zeros.* With the additional assumption that all zeros of the polynomial g are real, this was proved by Prüfer [23, Ch. V, Problem 182]. Corollary 1.1 also follows from the result of Bergweiler and Fuchs [3].

For the early history of results on the conjectures of Wiman and Pólya we refer to [11, 18], which contain ample bibliography. The main result of Levin and Ostrovskii [18] is

Theorem B *If f is a real entire function and all zeros of ff'' are real then*

$$\log^+ \log^+ |f(z)| = O(|z| \log |z|), \quad z \to \infty. \tag{1}$$

This shows that a function satisfying the assumptions of Theorem 1.1 cannot grow too fast, but there is a gap between Theorem B and Theorem A. Our Theorem 1.2 bridges this gap.

One important tool brought by Levin and Ostrovskii to the subject was a factorization of the logarithmic derivative of a real entire function f with only real zeros:

$$\frac{f'}{f} = \psi\phi,$$

where ϕ is a real entire function, and ψ is either identically 1 or a meromorphic function which maps the upper half-plane $H = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$ into itself. This factorization was used in all subsequent work in the subject. A standard estimate for analytic functions mapping the upper half-plane into itself shows that ψ is neither too large nor too small away from the real axis, so the asymptotic behaviour of f'/f mostly depends on that of ϕ. One can show that f is of finite order if and only if ϕ is a polynomial.

The second major contribution of Levin and Ostrovskii was the application of ideas from the value distribution theory of meromorphic functions in a half-plane [7]. (An earlier application of the value distribution theory to these questions is due to Edrei [5]). Using Nevanlinna theory, Hayman [8] proved that for an entire function f, the condition $f(z)f''(z) \neq 0,\ z \in \mathbb{C}$, implies that f'/f is constant. The assumptions of Theorem B mean that $f(z)f''(z) \neq 0$ in H. Levin and Ostrovskii adapted Hayman's argument to functions in a half-plane to produce an estimate for the logarithmic derivative. An integration of this estimate gives (1). To estimate the logarithmic derivative using Hayman's argument they applied a counterpart of the Nevanlinna characteristic for meromorphic functions in a half-plane, and proved an analogue of the main technical result of Nevanlinna theory, the lemma on the logarithmic derivative. This characteristic has two independent origins, [16] and [25], and the name "Tsuji characteristic" was introduced in [18].

In this paper we use both main ingredients of the work of Levin and Ostrovskii, the factorization of f'/f and the Tsuji characteristic.

Another important tool comes from Sheil-Small's proof of Theorem A. His key idea was the study of topological properties of the auxiliary function

$$F(z) = z - \frac{f(z)}{f'(z)}.$$

In the last section of his paper, Sheil-Small discusses the possibility of extension of his method to functions of infinite order, and proves the fact which turns out to be crucial: *if f is a real entire function, ff'' has only real zeros, and f' has a non-real zero, then F has a non-real asymptotic value.* In §4 we prove a generalization of this fact needed in our argument.

The auxiliary function F appears when one solves the equation $f(z) = 0$ by Newton's method. This suggests the idea of iterating F and using the Fatou–Julia theory of iteration of meromorphic functions. This was explored by Eremenko and Hinkkanen (see, for example, [13]).

Theorem 1.2 will be proved by establishing a more general result conjectured by Sheil-Small [24]. Let L be a meromorphic function in the plane, real on the real axis, such that all but finitely many poles of L are real and simple and have positive residues. Then L has a Levin–Ostrovskii representation [11, 18, 24]

$$L = \psi\phi \tag{2}$$

in which:

(a) ψ and ϕ are meromorphic in the plane and real on the real axis;
(b) ψ maps the upper half-plane into itself, or $\psi \equiv 1$;
(c) every pole of ψ is real and simple and is a simple pole of L;
(d) ϕ has finitely many poles.

We outline how such a factorization (2) is obtained. If L has finitely many poles, set $\psi = 1$. Assuming next that L has infinitely many poles, let

$$\ldots < a_{k-1} < a_k < a_{k+1} < \ldots$$

be the sequence of real poles of L enumerated in increasing order. Then for $|k|$ large, a_k and a_{k+1} are real and of the same sign, and are both simple poles of L with positive residue, so that there is at least one zero of L in the interval (a_k, a_{k+1}). We choose one such zero in each such interval and denote it by b_k. Then we set, for some large k_0,

$$\psi(z) = \prod_{|k| \geq k_0} \frac{1 - z/b_k}{1 - z/a_k},$$

and the product converges since the series

$$\sum_{|k| \geq k_0} \left(\frac{1}{a_k} - \frac{1}{b_k} \right)$$

converges by the alternating series test. For $\operatorname{Im} z > 0$ we then have $0 < \sum_{|k| \geq k_0} \arg \frac{b_k - z}{a_k - z} < \pi$ and so $\operatorname{Im} \psi(z) > 0$. Finally, we define ϕ by (2), and properties (a)–(d) follow (for the details see [11, 18, 24]).

Theorem 1.3 *Let L be a function meromorphic in the plane, real on the real axis, such that all but finitely many poles of L are real and simple and have positive residues. Let ψ, ϕ be as in* (2) *and* (a), (b), (c), (d). *If ϕ is transcendental then $L + L'/L$ has infinitely many non-real zeros.*

Theorem 1.3 is proved in §3 and §4, while Theorem 1.2 is deduced from Theorem 1.3 in §5.

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2 Preliminaries

We will require the following well known consequence of Carleman's estimate for harmonic measure.

Lemma 2.1 *Let u be a non-constant continuous subharmonic function in the plane. For $r > 0$ let $B(r, u) = \max\{u(z) : |z| = r\}$, and let $\theta(r)$ be the angular measure of that subset of the circle $C(0, r) = \{z \in \mathbb{C} : |z| = r\}$ on which $u(z) > 0$. Define $\theta^*(r)$ by $\theta^*(r) = \theta(r)$, except that $\theta^*(r) = \infty$ if $u(z) > 0$ on the whole circle $C(0, r)$. Then if $r > 2r_0$ and $B(r_0, u) > 1$ we have*

$$\log \|u^+(4re^{i\theta})\| \geq \log B(2r, u) - c_1 \geq \int_{2r_0}^{r} \frac{\pi dt}{t\theta^*(t)} - c_2,$$

in which c_1 and c_2 are absolute constants, and

$$\|u^+(re^{i\theta})\| = \frac{1}{2\pi} \int_{-\pi}^{\pi} \max\{u(re^{i\theta}), 0\} \, d\theta.$$

The first inequality follows from Poisson's formula, and for the second we refer to [26, Thm III.68]. Note that in the case that $u = \log |f|$ where f is an entire function, $\|u^+(re^{i\theta})\|$ coincides with the Nevanlinna characteristic $T(r, f)$.

Next, we need the characteristic function in a half-plane as developed by Tsuji [25] and Levin and Ostrovskii [18] (see also [7] for a comprehensive treatment). Let g be a meromorphic function in a domain containing the closed upper half-plane $\overline{H} = \{z \in \mathbb{C} : \operatorname{Im} z \geq 0\}$ (this hypothesis can be weakened [18]). For $t \geq 1$ let $\mathfrak{n}(t, g)$ be the number of poles of g, counting multiplicity, in $\{z : |z - it/2| \leq t/2, |z| \geq 1\}$, and set

$$\mathfrak{N}(r, g) = \int_1^r \frac{\mathfrak{n}(t, g)}{t^2} dt, \quad r \geq 1.$$

The Tsuji characteristic is defined as

$$\mathfrak{T}(r, g) = \mathfrak{m}(r, g) + \mathfrak{N}(r, g),$$

where

$$\mathfrak{m}(r, g) = \frac{1}{2\pi} \int_{\sin^{-1}(1/r)}^{\pi - \sin^{-1}(1/r)} \frac{\log^+ |g(r \sin\theta e^{i\theta})|}{r \sin^2 \theta} d\theta.$$

The upper half-plane is thus exhausted by circles of diameter $r \geq 1$ tangent to the real axis at 0. For non-constant g and any $a \in \mathbb{C}$ the first fundamental theorem then reads [7, 25]

$$\mathfrak{T}(r,g) = \mathfrak{T}(r, 1/(g-a)) + O(1), \quad r \to \infty, \tag{3}$$

and the lemma on the logarithmic derivative [18, p. 332], [7, Theorem 5.4] gives

$$\mathfrak{m}(r, g'/g) = O(\log r + \log^+ \mathfrak{T}(r,g)) \tag{4}$$

as $r \to \infty$ outside a set of finite measure. Further, $\mathfrak{T}(r,g)$ differs from a non-decreasing function by a bounded additive term [25]. Standard inequalities give

$$\mathfrak{T}(r, g_1+g_2) \leq \mathfrak{T}(r,g_1)+\mathfrak{T}(r,g_2)+\log 2, \quad \mathfrak{T}(r, g_1g_2) \leq \mathfrak{T}(r,g_1)+\mathfrak{T}(r,g_2), \tag{5}$$

whenever g_1, g_2 are meromorphic in $\overline{H}$. Using the obvious fact that $\mathfrak{T}(r, 1/z) = 0$ for $r \geq 1$ we easily derive from (3) and (5) that $\mathfrak{T}(r,g)$ is bounded if g is a rational function. (A more general result from [25] is that $\mathfrak{T}(r,g)$ is bounded if and only if g is a ratio of two bounded holomorphic functions in H.)

A key role will be played by the following two results from [18]. The first is obtained by a change of variables in a double integral [18, p. 332].

Lemma 2.2 *Let $Q(z)$ be meromorphic in $\overline{H}$, and for $r \geq 1$ set*

$$m_{0\pi}(r,Q) = \frac{1}{2\pi} \int_0^\pi \log^+ |Q(re^{i\theta})| d\theta. \tag{6}$$

Then for $R \geq 1$ we have

$$\int_R^\infty \frac{m_{0\pi}(r,Q)}{r^3} dr \leq \int_R^\infty \frac{\mathfrak{m}(r,Q)}{r^2} dr. \tag{7}$$

The second result from [18] is the analogue for the half-plane of Hayman's Theorem 3.5 from [9].

Lemma 2.3 *Let $k \in \mathbb{N}$ and let g be meromorphic in $\overline{H}$, with $g^{(k)} \not\equiv 1$. Then*

$$\begin{aligned} \mathfrak{T}(r,g) \quad \leq \quad & \left(2+\frac{1}{k}\right) \mathfrak{N}\left(r, \frac{1}{g}\right) + \left(2+\frac{2}{k}\right) \mathfrak{N}\left(r, \frac{1}{g^{(k)}-1}\right) + \\ & O(\log r + \log^+ \mathfrak{T}(r,g)) \end{aligned}$$

as $r \to \infty$ outside a set of finite measure.

Lemma 2.3 is established by following Hayman's proof exactly, but using the Tsuji characteristic and the lemma on the logarithmic derivative (4).

We also need the following result of Yong Xing Gu (Ku Yung-hsing, [14]).

Lemma 2.4 *For every $k \in \mathbb{N}$, the meromorphic functions g in an arbitrary domain with the properties that $g(z) \neq 0$ and $g^{(k)}(z) \neq 1$ form a normal family.*

A simplified proof of this result is now available [27]. It is based on a rescaling lemma of Zalcman–Pang [20] which permits an easy derivation of Lemma 2.4 from the following result of Hayman: *Let $k \in \mathbb{N}$ and let g be a meromorphic function in the plane such that $g(z) \neq 0$ and $g^{(k)}(z) \neq 1$ for $z \in \mathbb{C}$. Then* $g = \text{const}$, see [8] or [9, Corollary of Thm 3.5].

3 Proof of Theorem 1.3

Let L, ψ, ϕ be as in the hypotheses, and assume that ϕ is transcendental but $L + L'/L$ has only finitely many non-real zeros. Condition (b) implies the Carathéodory inequality:

$$\frac{1}{5}|\psi(i)|\frac{\sin\theta}{r} < |\psi(re^{i\theta})| < 5|\psi(i)|\frac{r}{\sin\theta}, \quad r \geq 1, \quad \theta \in (0, \pi), \tag{8}$$

see, for example, [17, Ch. I.6, Thm 8′].

Lemma 3.1 *The Tsuji characteristic of L satisfies $\mathfrak{T}(r, L) = O(\log r)$ as $r \to \infty$.*

Proof. We apply Lemma 2.3 almost exactly as in [18, p. 334]. Let $g_1 = 1/L$. Then

$$g_1' = -L'/L^2.$$

Since L has finitely many non-real poles and since $L + L'/L$ has by assumption finitely many non-real zeros it follows that g_1 and $g_1' - 1$ have finitely many zeros in H. Lemma 2.3 with $k = 1$ now gives $\mathfrak{T}(r, g_1) = O(\log r)$ initially outside a set of finite measure, and hence without exceptional set since $\mathfrak{T}(r, g_1)$ differs from a non-decreasing function by a bounded term. Now apply (3). □

Remark. The condition that L has finitely many non-real poles in Lemma 3.1 can be replaced by a weaker condition that $\mathfrak{N}(r, L) = O(\log r)$, $r \to \infty$, without changing the statement of the lemma or its proof.

Since ϕ has finitely many poles and is real on the real axis there exist a real entire function ϕ_1 and a rational function R_1 with

$$\phi = \phi_1 + R_1, \quad R_1(\infty) = 0. \tag{9}$$

Lemma 3.2 *The entire function ϕ_1 has order at most* 1.

Proof. Again, this proof is almost identical to the corresponding argument in [18]. Lemmas 2.2 and 3.1 give

$$\int_R^\infty \frac{m_{0\pi}(r,L)}{r^3}dr \leq \int_R^\infty \frac{\mathfrak{m}(r,L)}{r^2}dr = O(R^{-1}\log R), \quad R \to \infty.$$

Since $m_{0\pi}(r, 1/\psi) = O(\log r)$ by (8), we obtain using (2)

$$\int_R^\infty \frac{m_{0\pi}(r,\phi)}{r^3}dr = O(R^{-1}\log R), \quad R \to \infty.$$

But ϕ_1 is entire and real on the real axis and so

$$\| \log^+ |\phi_1(re^{i\theta})| \| = 2m_{0\pi}(r, \phi_1) \leq 2m_{0\pi}(r,\phi) + O(1),$$

using (9). Since $\| \log^+ |\phi_1(re^{i\theta})| \|$ is a non-decreasing function of r we deduce that

$$\| \log^+ |\phi_1(Re^{i\theta})| \| = O(R \log R), \quad R \to \infty,$$

which proves the lemma. □

Lemma 3.3 *Let $\delta_1 > 0$ and $K > 1$. Then we have*

$$|wL(w)| > K, \quad |w| = r, \quad \delta_1 \leq \arg w \leq \pi - \delta_1, \tag{10}$$

for all r outside a set E_1 of zero logarithmic density.

Proof. Choose δ_2 with $0 < \delta_2 < \delta_1$. Let

$$\Omega_0 = \{z \in \mathbb{C} : \frac{1}{2} < |z| < 2, \quad \frac{\delta_2}{2} < \arg z < \pi - \frac{\delta_2}{2}\}.$$

For $r \geq r_0$, with r_0 large, let $g_r(z) = 1/(rL(rz))$. Then $g_r(z) \neq 0$ on Ω_0, provided r_0 is large enough, since all but finitely many poles of L are real. Further,

$$g_r'(z) = -L'(rz)/L(rz)^2.$$

Since L has finitely many poles in H and $L + L'/L$ has finitely many zeros in H it follows that provided r_0 is large enough the equation $g_r'(z) = 1$ has no solutions in Ω_0. Thus the functions $g_r(z)$ form a normal family on Ω_0, by Lemma 2.4 with $k = 1$.

Suppose that $|w_0| = r \geq r_0$, and $\delta_1 \leq \arg w_0 \leq \pi - \delta_1$, and that

$$|w_0 L(w_0)| \leq K. \tag{11}$$

Then

$$|g_r(z_0)| \geq 1/K, \quad z_0 = \frac{w_0}{r},$$

and so since the g_r are zero-free and form a normal family we have

$$|g_r(z)| \geq 1/K_1, \quad |z| = 1, \quad \delta_2 \leq \arg z \leq \pi - \delta_2, \tag{12}$$

for some positive constant $K_1 = K_1(r_0, \delta_1, \delta_2, K)$, independent of r. We may assume that r_0 is so large that $|R_1(z)| \leq 1$ for $|z| \geq r_0$, in which the rational function R_1 is as defined in (9). By (2), (8), (9) and (12) we have, for $|w| = r$, $\delta_2 \leq \arg w \leq \pi - \delta_2$, the estimates

$$\begin{gathered} |wL(w)| = |w\psi(w)\phi(w)| \leq K_1, \\ |\phi_1(w)| \leq 1 + |\phi(w)| \leq K_2 = 1 + \frac{5K_1}{|\psi(i)| \sin \delta_2}. \end{gathered} \tag{13}$$

Thus (11) implies (13). For $t \geq r_0$ let

$$E_2(t) = \{w \in \mathbb{C} : |w| = t, |\phi_1(w)| > K_2\}.$$

Further, let $\theta(t)$ be the angular measure of $E_2(t)$, and as in Lemma 2.1 let $\theta^*(t) = \theta(t)$, except that $\theta^*(t) = \infty$ if $E_2(t) = C(0, t)$. Let

$$E_3 = \{t \in [r_0, \infty) : \theta(t) \leq 4\delta_2\}.$$

Since (11) implies (13), we have (10) for $t \in [r_0, \infty) \setminus E_3$. Applying Lemma 2.1 we obtain, since ϕ_1 has order at most 1 by Lemma 3.2,

$$(1 + o(1)) \log r \geq \int_{r_0}^{r} \frac{\pi dt}{t\theta^*(t)} \geq \int_{[r_0, r] \cap E_3} \frac{\pi dt}{4\delta_2 t},$$

from which it follows that E_3 has upper logarithmic density at most $4\delta_2/\pi$. Since δ_2 may be chosen arbitrarily small, the lemma is proved. $\square$

The estimates (8) and (10) and the fact that ϕ is real now give

$$|\phi(z)| > \frac{K \sin \delta_1}{5|\psi(i)|r^2}, \quad \delta_1 \leq |\arg z| \leq \pi - \delta_1,$$

for $|z| = r$ in a set of logarithmic density 1. Since ϕ has order at most 1 by (9) and Lemma 3.2, but is transcendental with finitely many poles, we deduce:

Lemma 3.4 *The function ϕ has infinitely many zeros.* □

Lemma 3.5 *There exist infinitely many zeros $\eta \in H \cup \mathbb{R}$ of L which satisfy at least one of the following conditions:*
(I) $\eta \in H$,
(II) $L'(\eta) = 0$,
(III) $\eta \in \mathbb{R}$ *and* $L'(\eta) > 0$.

Proof. By Lemma 3.4 ϕ has infinitely many zeros; by the hypotheses (2) and (c) these must be zeros of L. We assume that there are only finitely many zeros of L satisfying (I) or (II) and deduce that there are infinitely many zeros with the property (III).

Let $\{a_k\}$ be the real poles of L, in increasing order. By (2) and (d) all but finitely many of these are poles of ψ. Then there are two possibilities.

The first is that there exist infinitely many intervals (a_k, a_{k+1}) each containing at least one zero x_k of ϕ. Then we may assume that a_k and a_{k+1} are poles of ψ, with negative residues using (b). Hence there must be a zero y_k of ψ in (a_k, a_{k+1}), and we may assume that $y_k \neq x_k$, since L has by assumption finitely many multiple zeros. But then the graph of L must cut the real axis at least twice in (a_k, a_{k+1}), and so there exists a zero η of L in (a_k, a_{k+1}) with $L'(\eta) > 0$. Thus we obtain (III).

The second possibility is that we have infinitely many pairs of zeros a, b of ϕ such that L has no poles on $[a, b]$. In this case we again obtain a zero η of L with $L'(\eta) > 0$, this time in $[a, b]$, and again we have (III). □

Let

$$F(z) = z - \frac{1}{L(z)}, \quad F'(z) = 1 + \frac{L'(z)}{L(z)^2}. \tag{14}$$

Since L has finitely many non-real poles and $L + L'/L$ has finitely many non-real zeros we obtain at once:

Lemma 3.6 *The function F has finitely many critical points over $\mathbb{C} \setminus \mathbb{R}$, i.e. zeros z of F' with $F(z)$ non-real.* $\square$

Lemma 3.7 *There exists $\alpha \in H$ with the property that $F(z) \to \alpha$ as $z \to \infty$ along a path γ_α in H.*

Lemma 3.7 is a refinement of Theorem 4 of [24], and will be proved in §4.

Now set

$$g(z) = z^2 L(z) - z = \frac{zF(z)}{z - F(z)}, \quad h(z) = \frac{1}{F(z) - \alpha}, \tag{15}$$

in which α is as in Lemma 3.7. Then g has finitely many poles in H and (5), (14) and Lemma 3.1 give

$$\mathfrak{T}(r, g) + \mathfrak{T}(r, h) = O(\log r), \quad r \to \infty.$$

Hence Lemma 2.2 leads to

$$\int_1^\infty \frac{m_{0\pi}(r, g)}{r^3} dr + \int_1^\infty \frac{m_{0\pi}(r, h)}{r^3} dr < \infty, \tag{16}$$

in which $m_{0\pi}(r, g)$ and $m_{0\pi}(r, h)$ are as defined in (6).

Lemma 3.8 *The function F has at most four finite non-real asymptotic values.*

Proof. Assume the contrary. Since $F(z)$ is real on the real axis we may take distinct finite non-real $\alpha_0, \ldots, \alpha_n$, $n \geq 2$, such that $F(z) \to \alpha_j$ as $z \to \infty$ along a simple path $\gamma_j : [0, \infty) \to H \cup \{0\}$. Here we assume that $\gamma_j(0) = 0$, that $\gamma_j(t) \in H$ for $t > 0$, and that $\gamma_j(t) \to \infty$ as $t \to \infty$. We may further assume that $\gamma_j(t) \neq \gamma_{j'}(t')$ for $t > 0$, $t' > 0$, $j \neq j'$.

Re-labelling if necessary, we obtain n pairwise disjoint simply connected domains $D_1, \ldots, D_n$ in H, with D_j bounded by γ_{j-1} and γ_j, and for $t > 0$ we let $\theta_j(t)$ be the angular measure of the intersection of D_j with the circle $C(0, t)$. Since g has finitely many poles in H there exists a rational function R_2, with $R_2(\infty) = 0$, such that $g_2 = g - R_2$ is analytic in $H \cup \{0\}$. By (15), the function $g_2(z)$ tends to α_j as $z \to \infty$ on γ_j. Thus $g_2(z)$ is unbounded on each D_j but bounded on the finite boundary ∂D_j of each D_j.

Let c be large and positive, and for each j define

$$u_j(z) = \log^+ |g_2(z)/c|, \quad z \in D_j. \tag{17}$$

Set $u_j(z) = 0$ for $z \notin D_j$. Then u_j is continuous, and subharmonic in the plane since g_2 is analytic in $H \cup \{0\}$.

Lemma 2.1 gives, for some $R > 0$ and for each j,

$$\int_R^r \frac{\pi dt}{t\theta_j(t)} \leq \log \|u_j(4re^{i\theta})\| + O(1)$$

as $r \to \infty$. Since u_j vanishes outside D_j we deduce using (17) that

$$\int_R^r \frac{\pi dt}{t\theta_j(t)} \leq \log m_{0\pi}(4r, g_2) + O(1) \leq \log m_{0\pi}(4r, g) + O(1), \quad r \to \infty, \tag{18}$$

for all $j \in \{1, \ldots, n\}$. However, the Cauchy-Schwarz inequality gives

$$n^2 \leq \sum_{j=1}^n \theta_j(t) \sum_{j=1}^n \frac{1}{\theta_j(t)} \leq \sum_{j=1}^n \frac{\pi}{\theta_j(t)}$$

which on combination with (18) leads to, for some positive constant c_3,

$$n \log r \leq \log m_{0\pi}(4r, g) + O(1), \quad m_{0\pi}(r, g) \geq c_3 r^n, \quad r \to \infty.$$

Since $n \geq 2$ this contradicts (16), and Lemma 3.8 is proved. □

From Lemmas 3.6 and 3.8 we deduce that the inverse function F^{-1} has finitely many non-real singular values. Using Lemma 3.7, take $\alpha \in H$ such that $F(z) \to \alpha$ along a path γ_α tending to infinity in H, and take ε_0 with $0 < \varepsilon_0 < \operatorname{Im} \alpha$ such that F has no critical or asymptotic values in $0 < |w - \alpha| \leq \varepsilon_0$. Take a component C_0 of the set $\{z \in \mathbb{C} : |F(z) - \alpha| < \varepsilon_0\}$ containing an unbounded subpath of γ_α. Then by a standard argument [19, XI.1.242] involving a logarithmic change of variables the inverse function F^{-1} has a logarithmic singularity over α, the component C_0 is simply connected, and $F(z) \neq \alpha$ on C_0. Further, the boundary of C_0 consists of a single simple curve going to infinity in both directions. Thus we may define a continuous, non-negative, non-constant subharmonic function in the plane by

$$u(z) = \log \left| \frac{\varepsilon_0}{F(z) - \alpha} \right| = \log |\varepsilon_0 h(z)| \quad (z \in C_0), \quad u(z) = 0 \quad (z \notin C_0), \tag{19}$$

using (15).

The next lemma follows from (14) and (19).

Lemma 3.9 *For large z with $|zL(z)| > 3$ we have $|F(z) - \alpha| > |z|/2$ and $u(z) = 0$.* □

Lemma 3.10 *We have*

$$\lim_{r\to\infty} \frac{\log \|u(re^{i\theta})\|}{\log r} = \infty. \tag{20}$$

Proof. Apply Lemma 3.3, with $K = 3$ and δ_1 small and positive. By Lemma 3.9 we have $u(z) = 0$ if $\delta_1 \leq |\arg z| \leq \pi - \delta_1$ and $|z|$ is large but not in E_1. For large t let $\sigma(t)$ be the angular measure of that subset of $C(0,t)$ on which $u(z) > 0$. Since u vanishes on the real axis Lemma 2.1 and Lemma 3.3 give, for some $R > 0$,

$$\log \|u(4re^{i\theta})\| + O(1) \geq \int_R^r \frac{\pi dt}{t\sigma(t)} \geq \int_{[R,r]\setminus E_1} \frac{\pi dt}{4\delta_1 t} \geq \frac{\pi}{4\delta_1}(1 - o(1)) \log r$$

as $r \to \infty$. Since δ_1 may be chosen arbitrarily small the lemma follows. □

Now (19) gives

$$\|u(re^{i\theta})\| \leq m_{0\pi}(r, h) + O(1),$$

from which we deduce using (20) that

$$\lim_{r\to\infty} \frac{\log m_{0\pi}(r, h)}{\log r} = \infty.$$

This obviously contradicts (16), and Theorem 1.3 is proved. □

4 Proof of Lemma 3.7

The proof is based essentially on Lemmas 1 and 5 and Theorem 4 of [24]. Assume that there is no $\alpha \in H$ such that $F(z)$ tends to α along a path tending to infinity in H.

Let

$$W = \{z \in H : F(z) \in H\}, \quad Y = \{z \in H : L(z) \in H\}.$$

Then $Y \subseteq W$, by (14), so that each component C of Y is contained in a component A of W.

Lemma 4.1 *All but finitely many components C of Y are unbounded and satisfy*

$$\limsup_{z\to\infty,\, z\in C} \operatorname{Im} L(z) > 0. \tag{21}$$

Proof. Suppose first that C is a component of Y such that ∂C contains no pole of L. Then $\operatorname{Im} L(z)$ is harmonic and positive in C, and vanishes on ∂C. Thus C satisfies both conclusions of the lemma by the maximum principle.

Since each pole of L belongs to the closure of at most finitely many components C of Y, it suffices therefore to show that L has at most finitely many poles in the closure of Y. To see this, let x_0 be a pole of L, with $|x_0|$ large. Then x_0 is real, and is a simple pole of L with positive residue. Hence $\lim_{y\to 0+} \operatorname{Im} L(x_0+iy) = -\infty$ and since L is univalent on an open disc $N_0 = B(x_0, R_0)$ it follows that $\operatorname{Im} L(z) < 0$ on $N_0 \cap H$. Thus $N_0 \cap Y = \emptyset$. $\square$

Lemma 4.2 *To each component A of W corresponds a finite number $v(A)$ such that F takes every value at most $v(A)$ times in A and has at most $v(A)$ distinct poles on ∂A. Moreover, $v(A) = 1$ for all but finitely many components A of W.*

Proof. By Lemma 3.6, F has finitely many critical points in W, so only finitely many components A of the set W can contain critical points of F. Further, the assumption made in the beginning of this section implies that there is no $\alpha \in H$ such that $F(z)$ tends to α along a path tending to infinity in W.

Suppose first that A is a component of W which contains no critical points of F. Then every branch of F^{-1} with values in A can be analytically continued along every path in H. This implies that F maps A univalently onto H, and we set $v(A) = 1$ in this case.

Now consider a component A of W on which F is not univalent. Then A contains finitely many critical points of F, which we denote by $z_1, \ldots, z_p$. We connect the points $0, F(z_1), \ldots, F(z_p)$ by a simple polygonal curve $\Gamma \subset H \cup \{0\}$, so that the region $D = H\backslash\Gamma$ is simply connected. Let $X = \{z \in A : F(z) \in D\}$. Then every branch of F^{-1} with values in X can be analytically continued along every curve in D, so every component B of X is conformally equivalent to D via F.

If $\partial B \cap A$ contains no critical points of F then the inverse branch F_B^{-1} which maps D onto B can be analytically continued into H, so in this case

$F : A \to H$ is a conformal equivalence which contradicts our assumption that F is not univalent in A.

As every critical point of F can belong to the boundaries of only finitely many conponents B, we conclude that the set X has finitely many components. Denoting the number of these components by $v(A)$ we conclude from the open mapping theorem that F takes every value at most $v(A)$ times in A.

To show that F has at most $v(A)$ poles on ∂A, it is enough to note that if $z_0 \in \partial A$ is a pole of F, then for every neighbourhood N of z_0, F assumes in $N \cap A$ all sufficiently large values in H. □

Remark. Once it is established that F takes every value finitely many times in A, the Riemann–Hurwitz formula shows that one can take $v(A) = p+1$, where p is the number of critical points of F in A, counting multiplicity, but we don't use this observation.

Lemma 4.3 *There are infinitely many components A of W which satisfy all of the following conditions: (i) A contains a component C of Y; (ii) $\partial A \cap \partial C$ contains a zero of L; (iii) F is univalent on A; (iv) C is unbounded and satisfies* (21)*.*

Proof. We recall from Lemma 3.5 that L has infinitely many zeros η, satisfying one of the conditions (I), (II) or (III) of Lemma 3.5. Fix such a zero η. We are going to show that there exists a component C of Y such that $\eta \in \partial C$. We write

$$L(z) = (z-\eta)^m (ae^{i\theta} + O(|z-\eta|)), \quad z \to \eta, \quad a > 0, \quad \theta \in [-\pi, \pi). \quad (22)$$

Let t_0 be small and positive, and set

$$\zeta(t) = \eta + t \exp\left\{ \frac{i}{m} \left(\frac{\pi}{2} - \theta \right) \right\}, \quad t \in (0, t_0].$$

Then $\arg L(\zeta(t)) \to \pi/2$ as $t \to 0$, and so $L(\zeta(t)) \in H$ for $t \in (0, t_0]$ if t_0 is small enough. We claim that $\zeta(t) \in H$ for $t \in (0, t_0]$, provided t_0 is small enough. In Case (I) this evidently holds if $t_0 < \operatorname{Im} \eta$. In Case (II) we have $m \geq 2$ in (22) and, furthermore, we may assume in this case that $\eta \in \mathbb{R}$ (otherwise we have Case (I) again) and thus $\theta \in \{0, -\pi\}$ since L is a real function. Then $\arg(\zeta(t) - \eta) \in (0, 3\pi/4]$, and so $\zeta(t) \in H$. Finally, in Case (III) we have $m = 1$ and $\theta = 0$, and so $\arg(\zeta(t) - \eta) = \pi/2$. This proves our claim, and thus $\zeta((0,t_0]) \subset Y$. Let C be that component of Y which contains the curve $\zeta((0,t_0])$. Then $\eta \in \partial C$ since $\zeta(t) \to \eta$ as $t \to 0$.

Thus there are infinitely many zeros η of L that belong to the boundaries of components C of the set Y. As $F(\eta) = \infty$, by (14), and $Y \subseteq W$, we have $\eta \in \partial C \cap \partial A$, where A is a component of the set W containing C. Lemma 4.2 implies that infinitely many zeros η of L cannot belong to the boundary of the same component A, and thus there are infinitely many such components A. Finally, (iii) follows from Lemma 4.2 and (iv) from Lemma 4.1. □

We now complete the proof of Lemma 3.7. Applying Lemma 4.3 we obtain at least one zero η of L, with $\eta \in \partial A \cap \partial C$, in which A, C are components of W, Y respectively, satisfying $C \subseteq A$ and conditions (iii) and (iv) of Lemma 4.3. Since $F(\eta) = \infty$ by (14), it follows that for an arbitrarily small neighbourhood N of η, all values w of positive imaginary part and sufficiently large modulus are taken by F in $A \cap N$. Using (iii) we deduce that $F(z)$ is bounded as $z \to \infty$ in A. Now (14) gives $L(z) \to 0$ as $z \to \infty$ in A, and hence as $z \to \infty$ in C. This contradicts (21). □

5 Proof of Theorem 1.2

Suppose that f is a real entire function, and that f has finitely many non-real zeros. Then $L = f'/f$ has finitely many non-real poles, and all poles of L are simple and have positive residues. Thus L has a representation (2).

Lemma 5.1 *Suppose that ϕ is a rational function. Then f has finite order.*

Proof. Lemma 5.1 may be proved by modifying arguments of Levin-Ostrovskii [18, pp. 336-337] or of Hellerstein and Williamson [12, pp. 500-501] based on the residues of ψ, or by the following argument using the Wiman-Valiron theory [10]. Denote by $N(r)$ the central index of f. By [10, Theorems 10 and 12], provided r lies outside a set E_4 of finite logarithmic measure and $|z_0| = r$, $|f(z_0)| = M(r,f) = \max\{|f(z)| : |z| = r\}$, we have

$$\frac{f'(z)}{f(z)} = \frac{N(r)}{z}(1 + o(1)), \quad z = z_0 e^{it}, \quad t \in [-N(r)^{-2/3}, N(r)^{-2/3}].$$

This leads to

$$\int_0^{2\pi} |f'(re^{it})/f(re^{it})|^{5/6} dt \geq N(r)^{1/6} r^{-5/6}, \quad r \to \infty, \quad r \notin E_4.$$

Since ϕ is by assumption a rational function, (2) and (8) give

$$\int_0^{2\pi} |f'(re^{it})/f(re^{it})|^{5/6} dt = O(r^M), \quad r \to \infty,$$

for some positive M. We deduce that $N(r) = O(r^{6M+5})$ as $r \to \infty$, and thus f has finite order [10, (1.8) and Theorem 6]. This proves Lemma 5.1. □

Since $L + L'/L = f''/f'$, Theorem 1.2 now follows from Theorem 1.3 and Lemma 5.1.

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