Entire functions of slow growth whose Julia set coincides with the plane

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Abstract

We construct a transcendental entire function f with $J(f) = \mathbf{C}$ such that f has arbitrarily slow growth; that is, $\log|f(z)| \leq \phi(|z|)\log|z|$ for $|z| > r_0$, where ϕ is an arbitrary prescribed function tending to infinity.

For an entire function f we denote the Julia set by $J(f)$. By definition, it is the complement of the maximal open set $F(f)$, the set of normality, where the iterates f^n form a normal family.

While for polynomials the Julia set always has empty interior, for transcendental functions it may coincide with the whole complex plane $\mathbf{C}$. The first example with this property was given by Baker [1] and later Misiurewicz [16] showed that this is the case for the exponential function. There are several methods of constructing such examples (besides [1] and [16] we refer to [3, p. 74], [4, p. 155, p. 172], [7, p. 167-168] [8, p. 225], [9, p. 625], [10, p. 610], [12]) but none of them seems to be applicable to entire functions of arbitrarily slow growth, the main problem being to exclude the possibility of a wandering component of the set of normality where the iterates tend to infinity. That such a wandering component may indeed occur for functions of arbitrarily slow growth was shown by Baker [2] and Hinkkanen [11]. Notice that for entire functions of order less than 1/2 there is always a sequence of critical values tending to infinity (see [13, p. 1788]). This makes usual arguments for the proof of the absence of wandering domains hard to apply.

Theorem 1 *Let $t \mapsto \phi(t) : [0, \infty) \to [1, \infty)$ be an arbitrary increasing function tending to ∞ as $t \to \infty$. Then there exists an entire function f*

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and $r_0 > 0$ *with the properties* $J(f) = \mathbf{C}$ *and*

$$\log|f(z)| \le \phi(|z|)\log|z|, \quad |z| > r_0.$$

We use the following notation: $D(R) = \{z : |z| < R\}$, $\Delta(R) = \{z \in \mathbf{C} : |z| > R\}$ and $A(R, R') = \{z : R < |z| < R'\}$, where $0 < R < R' < \infty$. The sequence $(P^n(z))_{n=0}^{\infty}$ is called the P-orbit of the point z.

The proof of Theorem 1 is based on the following

Proposition 1 *Let* P *be a polynomial,* $P(0) = 0$, $P(1) = 1$, $\deg P \ge 2$.*Assume that the* P*-orbits of all the critical points of* P *tend to infinity. Let* $z_1, \ldots, z_{k-1} \in \mathbf{C}$ *and* $m_1, \ldots, m_{k-1} \in \mathbf{N}$ *and suppose that* $P^{m_j}(z_j) = 0$ *for* $1 \le j \le k-1$*. Let* $z_k \in \mathbf{C}$, $\epsilon > 0$ *and* $R > 0$ *be given.*

Then there exists a polynomial Q, $Q(0) = 0$, $Q(1) = 1$, *such that the* Q*-orbits of all the critical points of* Q *tend to infinity, and there exist* $z_1', \ldots, z_k' \in \mathbf{C}$ *and* $m_k \in \mathbf{N}$ *such that* $|z_j - z_j'| < \epsilon$ *and* $Q^{m_j}(z_j) = 0$ *for* $1 \le j \le k$*. Moreover,* $|P(z) - Q(z)| < \epsilon$ *for* $z \in D(R)$, $\deg Q = \deg P + 1$ *and if* $a_1, \ldots, a_d$ *are the zeros of* P*, then* Q *has a zero in each disk* $|z - a_j| < \epsilon$*, and a zero in* $\Delta(R)$*.*

For the proof of Proposition 1 we need the following two lemmas. In these lemmas, we shall use some concepts from the theory of quasiconformal (and quasiregular) maps; see [15] for a general introduction to quasiconformal maps, and [5, 6] for a discussion of their role in complex dynamics.

Lemma 1 *For every* $\delta > 0$ *and* $\hat{R} > 0$ *there exists* $\eta > 0$ *such that every quasiconformal homeomorphism* $\phi : \mathbf{C} \to \mathbf{C}$ *fixing* 0 *and* 1 *with Beltrami coefficient* $\|\mu\|_\infty < \eta$ *satisfies*

$$|\phi(z) - z| < \delta, \quad \textit{for} \quad z \in D(\hat{R}).$$

Proof. Assume that the lemma is incorrect. Then there is a sequence of quasiconformal homeomorphisms (ϕ_n), each fixing 0 and 1, such that the corresponding Beltrami coefficients μ_n satisfy $\|\mu_n\|_\infty \to 0$, but

$$|\phi_n(z_n) - z_n| \ge \delta > 0$$

for some $z_n \in D(\hat{R})$. As a family of quasiconformal maps with uniformly bounded distortion fixing 0 and 1 is normal [15, §II.5], we may assume that $\phi_n \to \phi$ as $n \to \infty$, uniformly on compacta in $\mathbf{C}$, and ϕ is a conformal homeomorphism. Our normalization implies that $\phi(z) = z$ and we obtain a contradiction. □

Lemma 2 *For every positive integer d and $\eta > 0$ there exists $\gamma \in (0, 1/2)$ with the following property:*

Let h_1 and h_2 be holomorphic functions in $A(r/2, 4r)$ such that $\|h_i\|_\infty < \gamma$, $i = 1, 2$. Then there exists a quasiregular local homeomorphism $\phi : A(r, 2r) \to \mathbf{C}$ with boundary values

$$\phi(z) = z^d(1 + h_1(z)), \quad |z| = r$$

and

$$\phi(z) = z^d(1 + h_2(z)), \quad |z| = 2r$$

and the Beltrami coefficient μ of ϕ satisfies $\|\mu\|_\infty < \eta$.

Proof. We define $h(z) := (2 - |z|/r)h_1(z) + (|z|/r - 1)h_2(z)$. This function is smooth in the ring $A(r, 2r)$ and has boundary values $h_1(z)$, $|z| = r$, and $h_2(z)$, $|z| = 2r$. The sup-norm of the derivative $Dh : A(r, 2r) \to \mathbf{R}^2$ tends to 0 when $\gamma \to 0$. Thus $\phi(z) := z^d(1 + h(z))$ has all the required properties when γ is small enough. □

Proof of Proposition 1. It follows from our hypotheses on the critical points of P that $J(P)$ is totally disconnected and $P^n(z) \to \infty$ for all $z \in \mathbf{C}\backslash J(P)$, see, e. g., [6, p. 67].

Let $d := \deg P$. Recall (see [6, p. 34] or [18, p. 63, p. 147]) that the limit

$$u := \lim_{n\to\infty} \frac{1}{d^n} \log |P^n| \tag{1}$$

exists uniformly on compacta in $\mathbf{C}\backslash J(P)$ and u is a positive harmonic function there, satisfying

$$u(z) \sim \log |z|, \quad z \to \infty. \tag{2}$$

If we extend u by setting $u(z) = 0$ for $z \in J(P)$ the resulting function is continuous, and we have $u(z) > 0$ if and only if $z \in \mathbf{C}\backslash J(P)$.

We may assume without loss of generality that $z_k \in \mathbf{C}\backslash J(f)$, because this can be achieved by a small shift of z_k, using that $J(f)$ is totally disconnected. Performing another small shift of z_k if necessary, we may also assume that

$$0 < u(z_k) \neq d^j u(c) \quad \text{for all} \quad c \in \text{crit}(P) \quad \text{and} \quad j \in \mathbf{Z}, \tag{3}$$

where $\text{crit}(P)$ denotes the set of critical points of P. It follows from (3) that there exists $\kappa > 0$ with the property

$$|d^n u(z_k) - d^j u(c)| > \kappa d^n \quad \text{for all} \quad c \in \text{crit}(P) \quad \text{and} \quad n \in \mathbf{N},\, j \in \mathbf{Z},$$

from which it follows in view of (1) that

$$\min_{j\in\mathbf{N}}\left|\log\frac{|P^n(z_k)|}{|P^j(c)|}\right|\to\infty\quad\text{as}\quad n\to\infty\quad\text{and}\quad c\in\text{crit}(P).\tag{4}$$

Similarly

$$\min_{0\le j<n}\frac{|P^n(z_k)|}{|P^j(z_k)|}\to\infty\quad\text{as}\quad n\to\infty.\tag{5}$$

We fix arbitrary $\delta>0$ and apply Lemma 1 for some $\hat{R}$ satisfying $\hat{R}\ge R+1$, $\hat{R}\ge 1+\max_{1\le j\le d}|a_j|$, $\hat{R}\ge 1+\max_{1\le j\le k}|z_j|$, and $\hat{R}\ge 1+\max_{|z|=R+1}|P(z)|$. Then, using η obtained from Lemma 1 and d, we apply Lemma 2 to obtain $\gamma\in(0,1/2)$.

Now we are going to find a large integer n so that the following conditons (6)-(11) are satisfied.

$$|P^n(z_k)|>\max\left\{\frac{4}{\gamma}(R+1),\ \frac{2e(d+1)}{a}\right\},\tag{6}$$

where $a:=\lim_{z\to\infty}z^{-d}P(z)$,

$$r:=\frac{\gamma|P^n(z_k)|}{4}>\frac{16}{\gamma},\tag{7}$$

$$|a^{-1}z^{-d}P(z)-1|<\gamma,\quad\text{for}\quad z\in\Delta(r/2),\tag{8}$$

$$\min_{j\in\mathbf{N}}\left|\log\frac{|P^n(z_k)|}{|P^j(c)|}\right|>\log\frac{4}{\gamma},\quad c\in\text{crit}(P),\tag{9}$$

$$\min_{0\le j<n}\frac{|P^n(z_k)|}{|P^j(z_k)|}>\frac{4}{\gamma},\tag{10}$$

and

$$\text{the }\ P-\text{orbits of all points }\ z_1\dots,z_{k-1}\ \text{ are contained in }\ D(r).\tag{11}$$

Conditions (9) and (10) can be satisfied in view of (4) and (5) respectively.

We define a quasiregular map $Q_1:\mathbf{C}\to\mathbf{C}$ in the following way:

$$Q_1(z)=P(z),\quad z\in D(r),\tag{12}$$

$$Q_1(z)=az^d\left(1-\frac{z}{P^n(z_k)}\right),\quad z\in\Delta(2r),\tag{13}$$

and in the annulus $A(r,2r)$ we interpolate using Lemma 2 with $h_1(z)=a^{-1}z^{-d}P(z)-1$ and $h_2(z)=-z/P^n(z_k)$. The conditions of Lemma 2 are satisfied in view of (7) and (8).

If $U := \Delta(2|P^n(z_k)|)$ then U is Q_1-invariant and all Q_1-orbits in U tend to infinity. The map Q_1 has the following properties:

(i) the Q_1-orbits of the critical points of Q_1 tend to infinity.
Indeed, the critical set of Q_1 consists of the critical set of P and one additional point $w := dP^n(z_k)/(d+1)$. The P-orbits of the critical points of P do not intersect the annulus $A(r, 2|P^n(z_k)|)$ in view of (9), so their Q_1-orbits also do not intersect this annulus, but do intersect the set U, and thus tend to infinity. Furthermore, $Q_1(w) = aw^d/(d+1) \in U$ in view of (6), so the Q_1-orbit of w also tends to infinity.

(ii) $(Q_1)^{n+1}(z_k) = 0$.
Indeed, $(Q_1)^2((Q_1)^{n-1}(z_k)) = (Q_1)^2(P^{n-1}(z_k)) = Q_1(P^n(z_k)) = 0$, because $P^j(z_k) \in D(r)$ for $j < n$ in view of (10) and $P(z) = Q_1(z)$ for $z \in D(r)$ by definition.

(iii) $Q_1^{m_j}(z_j) = 0$ for $1 \leq j \leq k-1$.
This follows from (11) since $Q_1(z) = P(z)$ for $z \in D(r)$.

Thus Q_1 has all the required properties, except that it is not holomorphic in the annulus $A(r, 2r)$. To make it holomorphic we use a method of M. Shishikura [17]; see also [5, §§8-9] for an account of Shishikura's method. The image of the annulus $A(r, 2r)$ is contained in the invariant domain U, which is disjoint from $A(r, 2r)$. This permits us to define a new conformal structure σ in $\mathbf{C}$ such that it coincides with the standard conformal structure σ_0 in U, and $Q_1 : (\mathbf{C}, \sigma) \to (\mathbf{C}, \sigma)$ is holomorphic. The distortion of this structure with respect to the standard one is measured by the sup-norm of the Beltrami coefficient which is the same as that of Q_1, namely at most η (see Lemmas 1 and 2). By the basic existence theorem for quasiconformal mappings [15, Chapter 5], there exists a conformal homeomorphism $\psi : (\mathbf{C}, \sigma_0) \to (\mathbf{C}, \sigma)$. We can normalize it by $\psi(0) = 0$ and $\psi(1) = 1$. Then $Q := \psi^{-1} \circ Q_1 \circ \psi$ is easily seen to be a polynomial. The dynamics of Q are similar to those of Q_1, namely from (i)-(iii) it follows that the Q-orbits of the critical points of Q tend to infinity, and with $z'_j := \psi^{-1}(z_j),\ \ 1 \leq j \leq k$, and $m_k = n+1$ we have $Q^{m_j}(z'_j) = 0$ for $1 \leq j \leq k$.

Finally we notice that $\psi : (\mathbf{C}, \sigma_0) \to (\mathbf{C}, \sigma_0)$ is quasiconformal and the sup-norm of its Beltrami coefficient is at most η. The same is true for ψ^{-1} and so by Lemma 1 we have

$$|\psi(z) - z| < \delta \quad \text{and} \quad |\psi^{-1}(z) - z| < \delta \quad \text{for} \quad z \in D(\hat{R}).$$

If $\delta < 1$ and $|z| \leq R$, then $|\psi(z)| \leq R + \delta < r$ and hence $|Q_1(\psi(z))| =$

$|P(\psi(z))| \leq \hat{R} - 1$. We deduce that if $\delta \to 0$, then

$$Q(z) = \psi^{-1}(Q_1(\psi(z))) = \psi^{-1}(P(\psi(z))) \to P(z),$$

uniformly for $z \in D(R)$. This implies that Q and z'_j have all the required properties for sufficiently small δ. □

Proof of Theorem 1. We fix a dense sequence $(z_j)_{j=1}^{\infty}$ in $\mathbf{C}$ with $z_1 = 3/4$, a sequence of positive numbers (ϵ_j) with the property

$$\sum_{j=1}^{\infty} \epsilon_j < 1, \tag{14}$$

and an increasing sequence $(R_j) \to \infty$ with the property

$$\sum_{j=1}^{\infty} \frac{1}{R_j} < \infty. \tag{15}$$

Starting with $k = 2$, $P_2(z) = 4z^2 - 3z$, $m_1 = 1$ and $z_1 = z_{1,2} = 3/4$, we apply Proposition 1 repeatedly, and obtain a sequence (P_k) of polynomials and a sequence (m_k) of positive integers with the following properties: $\deg P_k = k$, $P_k(0) = 0$, $P_k(1) = 1$, and for every $j \in \mathbf{N}$ and $k > j$, there is a point $z_{j,k}$ satisfying

$$|z_k - z_{k,k+1}| < \epsilon_{k+1} \quad \text{and} \quad |z_{j,k} - z_{j,k+1}| < \epsilon_{k+1} \quad \text{for} \quad j < k$$

such that

$$P_k^{m_j}(z_{j,k}) = 0. \tag{16}$$

In addition, the zeros $a_{j,k}$ of P_k satisfy

$$|a_{k,k}| > R_k \quad \text{for} \quad k \geq 3 \quad \text{and} \quad |a_{j,k} - a_{j,k+1}| < \epsilon_{k+1} \quad \text{for} \quad k \geq 2, j \leq k,$$

and the sequence (P_k) converges uniformly on compacta in $\mathbf{C}$ to an entire function f.

It follows that the limits $w_j := \lim_{k\to\infty} z_{j,k}$ exist for all $j \in \mathbf{N}$ and $|z_j - w_j| \to 0$ as $j \to \infty$. Thus the sequence (w_j) is dense in $\mathbf{C}$. Passing to the limit as $k \to \infty$ in (16), we conclude that $f^{m_j}(w_j) = 0$. This means that the preimages of zero are dense in $\mathbf{C}$. Thus $J(f) = \mathbf{C}$.

Finally we have to estimate the growth. We have

$$P_k(z+1) = \prod_{j=1}^{k} \left(1 - \frac{z}{c_{j,k}}\right),$$

with $c_{j,k} = a_{j,k} - 1$. Thus $|c_{j,k} - c_{j,k+1}| < \epsilon_{k+1}$ for $k \geq 2, j \leq 2$ and $|c_{k,k}| > R_k - 1$ for $k \geq 3$. Passing to the limit when $k \to \infty$ and taking (15) into account we conclude that

$$f(z+1) = \prod_{j=1}^{\infty} \left(1 - \frac{z}{c_j}\right)$$

where $|c_j| = |\lim_{k\to\infty} c_{j,k}| > R_j - 1 - \sum_{n=j+1}^{\infty} \epsilon_n > R_j - 2$. Thus f is an entire function of genus zero. Using standard estimates for canonical products (see, for example, [14]) we can choose (R_j) so that the growth of f is arbitrarily slow.

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References

[1] I. N. Baker, Limit functions and sets of non-normality in iteration theory, Ann. Acad. Sci. Fenn. 467 (1970).

[2] I. N. Baker, Some entire functions with multiply-connected wandering domains, Ergodic Theory Dynamical Systems 5 (1985), 163-169.

[3] I. N. Baker, P. J. Rippon, Iteration of exponential functions, Ann. Acad. Sci. Fenn. 9 (1984), 49-77.

[4] W. Bergweiler, Iteration of meromorphic functions, Bull. Amer. Math. Soc. (N.S.) 29 (1993), 151-188.

[5] A. F. Beardon, Iteration of rational functions, Springer, New York 1991.

[6] L. Carleson, T. W. Gamelin, Complex dynamics, Springer, New York 1993.

[7] R. L. Devaney, Julia sets and bifurcation diagrams for exponential maps, Bull. Amer. Math. Soc. (N.S.) 11 (1984), 167-171.

[8] R. L. Devaney, Dynamics of entire maps, in: Dynamical Systems and Ergodic Theory, Banach Center Publ., vol. 23, Polish Scientific Publishers, Warsaw, 1989, pp. 221-228.

[9] A. Eremenko and M. Yu. Lyubich, The dynamics of analytic transforms, Leningrad Math. J. 1 (1990), 563-634.

[10] M. Herman, Are there critical points on the boundaries of singular domains? Comm. Math. Phys. 99 (1985), 593-612.

[11] A. Hinkkanen, On the size of Julia sets, in: Complex Analysis and its applications, Pitnam Research Notes in Mathematics Series 305 (1995), 178-189.

[12] C. M. Jang, Julia set of the function $z \exp(z + \mu)$, Tôhoku Math. J. 44 (1992), 271-277.

[13] J. K. Langley, On the multiple points of certain meromorphic functions, Proc. Amer. Math. Soc. 123 (1995), 1787-1795.

[14] B. Ya. Levin, Lectures on entire functions, Transl. Math. Monographs, vol. 150, AMS, Providence RI, 1996.

[15] O. Lehto, K. I. Virtanen, Quasiconformal mappings in the plane, Springer, Berlin - Heidelberg - New York 1973.

[16] M. Misiurewicz, On iterates of e^z, Ergodic Theory Dynamical Systems 1 (1981), 103-106.

[17] M. Shishikura, On the quasi-conformal surgery of rational functions, Ann. Sci. École Norm. Sup. (4) 20 (1987), 1-29.

[18] N. Steinmetz, Rational iteration, Walter de Gruyter, Berlin 1993.

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